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QUANTUM HOMOGENEOUS SPACES OF CONNECTED HOPF ALGEBRAS

KEN BROWN AND PAUL GILMARTIN

ABSTRACT. Let H be a connected Hopf k -algebra of finite Gel'fand-Kirillov dimension over an algebraically closed field k of characteristic 0. The objects of study in this paper are the left or right coideal subalgebras T of H . They are shown to be deformations of commutative polynomial k -algebras. A number of well-known homological and other properties follow immediately from this fact. Further properties are described, examples are considered, invariants are constructed and a number of open questions are listed.

1. INTRODUCTION

1.1. The left and right coideal subalgebras of a Hopf algebra H (defined in §2.2) have been an important focus of research since the classical work on the commutative and cocommutative cases in the last century, [10], [34], [29]. Following the seminal papers of Takeuchi [35], Masuoka [26] and Schneider [31] attention has concentrated on the *quantum homogeneous spaces*, that is those coideal subalgebras of H over which H is a faithfully flat module. This paper continues this line of research for the case where H is a connected Hopf k -algebra of finite Gel'fand-Kirillov dimension n , (written $\text{GKdim}H = n$), with k an algebraically closed field of characteristic 0.

This class of Hopf algebras has been the subject of a series of recent papers, see for example [40], [7], [37]. None of these, however, has examined their coideal subalgebras, so a primary aim here is to lay out their basic properties and clarify topics for future research. Motivating examples from the classical theory are plentiful and offer a rich source of intuition - thus, H as above is *commutative* if and only if it is the coordinate ring $\mathcal{O}(U)$ of a unipotent algebraic k -group U of dimension n , while H is *cocommutative* if and only if it is the enveloping algebra $U(\mathfrak{g})$ of its n -dimensional Lie k -algebra \mathfrak{g} of primitive elements. In the former case the coideal subalgebras of H are the (right and left) homogeneous U -spaces; in the latter case - thanks to cocommutativity - the coideal subalgebras are the Hopf subalgebras, hence are just the enveloping algebras of the Lie subalgebras of \mathfrak{g} . For references for these classical facts, see §3.3, §3.2.

1.2. In both the above classical settings the faithful flatness condition always holds, and both the Hopf algebra H and its coideal subalgebras either are themselves (in the first case), or have associated graded algebras which are (in the second case) commutative polynomial k -algebras. To extend this picture to the non-classical

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world, we need the concept of the *coradical filtration* of a Hopf algebra, recalled in §2.1. The starting point is then the result of Zhuang, [40, Theorem 6.9], restated below as Theorem 2.1, stating that the associated graded algebra $\text{gr}H$ of H with respect to its coradical filtration is a polynomial k -algebra in $n = \text{GKdim}H$ commuting variables. Our first main result shows that the whole of the above classical picture extends to the setting of Zhuang’s theorem.

Theorem 1.1. *Let H be a connected Hopf algebra of finite Gel’fand-Kirillov dimension n over an algebraically closed field k of characteristic 0 and let T be a left or right coideal subalgebra of H . Then*

- (1) (Masuoka, [26]) H is a free right and left T -module.
- (2) With respect to the coradical filtration of T , the associated graded algebra $\text{gr}T$ is a polynomial k -algebra in m variables.
- (3) $\text{GKdim}T = m \leq n$, and $m = n$ if and only if $T = H$.

The theorem is given below as Lemma 4.1 and Theorem 4.3. Just as with parts (1) and (2), so also (3) incorporates familiar classical phenomena: if $W \subset U$ is a strict inclusion of unipotent k -groups, then $\dim W < \dim U$; and a strict inclusion of Lie algebras of course implies a strict inequality of their dimensions.

A key point is that the polynomial algebra $\text{gr}H$ above is itself a Hopf algebra, the comultiplication and multiplication of H being “lifts” of those of $\text{gr}H$. In particular, $\text{gr}H$ is *connected*, and hence is the coordinate ring of a unipotent algebraic k -group U , as explained in §3.3. Moreover, T is a lift of the homogeneous U -space with coordinate ring $\text{gr}T$. The case where H is cocommutative, that is, H is isomorphic as a Hopf algebra to the enveloping algebra of an n -dimensional Lie algebra, is the case where U is abelian, that is, $U \cong (k, +)^n$.

1.3. Following [23], we call a left or right coideal subalgebra T of a Hopf algebra H a *quantum homogeneous space* if H is faithfully flat as a left *and* as a right T -module. Thus (1) of the theorem ensures that all coideal subalgebras of H as in the theorem are quantum homogeneous spaces.

As is completely standard, a filtered-graded result such as Theorem 1.1 has important homological consequences. Thus we deduce:

Corollary 1.2. *Let H and T be as in Theorem 1.1.*

- (1) T is a noetherian domain of global dimension m , AS-Regular and GK-Cohen-Macaulay .
- (2) T is twisted Calabi-Yau of dimension m .

For unexplained terminology in the above, see the references and definitions in §4.2, §4.3, where these results are proved. The twisting automorphism in (2) is discussed in the next subsection.

1.4. **The antipode.** As already mentioned, one of the pillars on which our work stands is the paper of Masuoka [26]. The relevant specialisation of the main result of that paper is stated here as Proposition 2.3, a central feature being the existence of a bijection between the left coideal subalgebras of H and the quotient right H -module coalgebras of H . Using this bijection and a well-known lemma due to Koppinen [17], we deduce the following, where (1) is well-known, and valid in a much wider setting (see Lemma 2.6), and the remaining parts are proved in Theorem 2.7 and Proposition 4.6.

Theorem 1.3. *Let H and T be as in Theorem 1.1, and let S denote the antipode of H .*

- (1) *The map $T \mapsto S(T)$ gives a bijection between the sets of right and left coideal subalgebras of H .*
- (2) *$S^2(T) = T$, and either $(S^2)|_T = \text{id}$ or $|(S^2)|_T| = \infty$.*
- (3) *Suppose T is a right quantum homogeneous space. Then the Nakayama automorphism of T is $S^2 \circ \tau_\chi^\ell$, where χ is the character of the right structure of the left integral of T . There is an analogous formula which applies when T is a left quantum homogeneous space.*

Both possibilities can occur in (2): for $T = H$, the smallest example with $|S|$ infinite occurs at Gel'fand-Kirillov dimension 3, namely the infinite family of examples $B(\lambda)$ found in [40] and recalled here in §3.7. For $T \neq H$, S^2 can have infinite order already at dimension 2, as we show (for a coideal subalgebra of $B(\lambda)$) in (3.3) in §3.5. For the unexplained terminology in (3), see §4.3. The determination of the Nakayama automorphism in (3) depends crucially on earlier work of Kraehmer [18] and of Liu and Wu [23], [24].

1.5. The signature and the lantern. Let H and T be as in Theorem 1.1. Given that $\text{gr}H$ and $\text{gr}T$ are graded polynomial algebras, their homogeneous generators have specific degrees whose multisets of values constitute invariants $\sigma(T)$ and $\sigma(H)$ of T [resp. H]. We call $\sigma(T)$ the *signature* of T , and write $\sigma(T) = (e_1^{(r_1)}, \dots, e_s^{(r_s)})$, where e_i and r_i are positive integers with $e_1 < e_2 < \dots < e_s$, and the term $e_i^{(r_i)}$ indicates that the degree e_i occurs r_i times among the graded polynomial generators of $\text{gr}T$. A closely related invariant is the *lantern* $\mathcal{L}(T)$ of T , defined in [37, Definition 1.2(d)] for H itself, extended here to a quantum homogeneous space T : namely, $\mathcal{L}(T)$ is the k -vector space of primitive elements of the graded dual $(\text{gr}(T))^*$. The basic properties of these invariants are as follows.

Theorem 1.4. *Let H be a connected Hopf algebra of finite Gel'fand-Kirillov dimension n over an algebraically closed field k of characteristic 0 and let T be a left coideal subalgebra of H with $\text{GKdim}T = m > 0$. Let $\sigma(H) = (d_1^{(m_1)}, \dots, d_t^{(m_t)})$ and $\sigma(T) = (e_1^{(r_1)}, \dots, e_s^{(r_s)})$. Then*

- (1) $\sum_i d_i m_i = n$ and $\sum_j e_j r_j = m$.
- (2) $d_1 = e_1 = 1$; if $n \geq 2$, then $r_1 \geq 2$.
- (3) $\sigma(H) = (1^{(n)})$ if and only if, as a Hopf algebra, $H \cong U(\mathfrak{g})$, the enveloping algebra of its n -dimensional Lie algebra \mathfrak{g} of primitive elements.
- (4) $\mathcal{L}(H) = \bigoplus_i \mathcal{L}(H)(d_i)$ is a positively graded Lie algebra of dimension n , with $\dim_k \mathcal{L}(H)(d_i) = m_i$.
- (5) $\mathcal{L}(H)$ is generated in degree 1 - that is, $\mathcal{L}(H) = \langle \mathcal{L}(H)(1) \rangle$.
- (6) If $i < t$ then $d_{i+1} = d_i + 1$.
- (7) $\mathcal{L}(T)$ is a graded quotient of $\mathcal{L}(H)$.
- (8) $\sigma(T)$ is a submultiset of $\sigma(H)$.

Parts (2),(3),(4), and (5) of this theorem were proved in [37], but they are proved again here in the course of proving the remaining parts, in §5. Finite dimensional positively graded (and therefore nilpotent) Lie algebras which are generated in degree 1, as $\mathcal{L}(H)$ is by (4) and (5) of the theorem, are called *Carnot Lie algebras* in the literature; see for example [9].

We view one important function of these equivalent invariants as being to provide a framework for future work on connected Hopf algebras. We discuss this aspect further in §6.

1.6. Examples. In §3 we describe what is known about various particular subclasses of quantum homogeneous spaces T of connected Hopf algebras H of Hopf algebras of finite Gel'fand-Kirillov dimension. Thus, we discuss the cases where H or T is commutative; where H or simply T is cocommutative; where $\text{GKdim}T \leq 2$; and where $\text{GKdim}H \leq 3$. Particularly noteworthy is the fact, recorded in Proposition 3.5 in §3.4, that

*the Jordan plane $J = k\langle X, Y : [X, Y] = Y^2 \rangle$ is a coideal subalgebra
of a connected Hopf algebra H with $\text{GKdim}H = 3$.*

It is not hard to see that J cannot be supplied with a comultiplication with respect to which it is a Hopf algebra.

1.7. Layout and notation. Some basic definitions, the keystone theorems of Masuoka and Zhuang, and the relevant properties of the antipode, are given in §2. Examples are discussed in §3; particularly crucial here is the commutative case, since this yields the required properties of $\text{gr}T$ and $\text{gr}H$. The homological results are described and proved in §4, and the signature and lantern are covered in §5. §6 contains a short discussion of future research directions.

Throughout let k denote a base field, which we assume to be algebraically closed and of characteristic 0, unless otherwise stated. All vector spaces and unadorned tensor products will be assumed to be over the base field k . For a Hopf algebra H , we use Δ , ϵ and S to denote the coproduct, counit and antipode respectively. The kernel of the counit will be denoted as H^+ . Throughout this paper, the bijectivity of the antipode S is incorporated as part of the definition of a Hopf algebra; in practice this makes no difference to the main results, since bijectivity of S always holds when H is connected by [26, Proposition 1.2].

2. BASIC DEFINITIONS

2.1. The Coradical Filtration. Let H be a Hopf algebra. Recall that an element $x \in H$ is *primitive* if $\Delta(x) = 1 \otimes x + x \otimes 1$. The subspace of primitive elements of H , denoted by $P(H)$, is called the *primitive space* of H ; $P(H)$ is a Lie subalgebra of H with respect to the commutator bracket $[x, y] = xy - yx$. Let H_0 denote the *coradical* of H , that is, the sum of all simple coalgebras of H , and define inductively the ascending chain of subcoalgebras of H ,

$$H_n := \Delta^{-1}(H \otimes H_{n-1} + H_0 \otimes H).$$

Then $\{H_n\}_{n=0}^\infty$ is the *coradical filtration* of H .

Suppose now that H is *connected*, that is, $H_0 = k$. Then $H_1 = k \oplus P(H)$, and, by [28, Lemma 5.2.8] $\{H_n\}_{n=0}^\infty$ forms an algebra as well as a coalgebra filtration of H , such that $S(H_i) \subseteq H_i$ for $i \geq 0$; that is, it is a *Hopf filtration*. Hence, the associated graded coalgebra of H with respect to the coradical filtration,

$$\text{gr} H = \bigoplus_{i=0}^{\infty} H(i) := \bigoplus_{i=0}^{\infty} H_i/H_{i-1}; \quad H_{-1} := \{0\}.$$

forms a graded connected Hopf algebra. Furthermore, by a result of Sweedler ([33, §11.2]) with several subsequent rediscoveries, when H is a connected Hopf algebra (over *any* field), $\text{gr } H$ is commutative.

As noted in [37, §1] and [40, Remark 5.5], if H is connected and $\dim_k(P(H)) < \infty$ (which is the case when $\text{GKdim } H < \infty$ since $U(P(H)) \subseteq H$), then $\dim H(i) < \infty$ for all $i \geq 0$. Thus, by [40, Theorem 6.9] (which we restate as Theorem 2.1 below), when H is connected of finite GK-dimension, $\text{gr } H$ is a graded polynomial algebra in finitely many homogeneous variables. Moreover, $\text{gr } H$ is a *coradically graded coalgebra* [2, Definition 1.13] - that is, for $n \geq 0$, $(\text{gr } H)_n = \bigoplus_{i=0}^n H(i)$. Summarising the above discussion, we obtain

Theorem 2.1. (Zhuang) [40, Theorem 6.9] *Assume that k is an algebraically closed field of characteristic 0 and let H be a connected Hopf k -algebra, with associated notation as introduced above. Then the following statements are equivalent:*

- (1) $\text{GKdim } H < \infty$.
- (2) $\text{GKdim } \text{gr } H < \infty$.
- (3) $\text{gr } H$ is finitely generated.
- (4) $\text{gr } H$ is isomorphic as an algebra to the polynomial k -algebra in ℓ variables for some $\ell \geq 0$.

In this case, $\text{GKdim } H = \text{GKdim } \text{gr } H = \ell$. Moreover, $\text{gr } H$ is connected both as a Hopf algebra and as a graded algebra.

Whenever we speak of the associated graded algebra of a connected Hopf algebra, we shall mean with respect to its coradical filtration. Similarly, the *degree* of an element of a Hopf algebra will always be understood to be with respect to the coradical filtration, unless otherwise stated.

2.2. Coideal subalgebras and quantum homogeneous spaces. Let H be a Hopf algebra. A subalgebra T of H is a *left coideal subalgebra* if

$$\Delta(T) \subseteq H \otimes T.$$

Similarly we say that T is a *right coideal subalgebra* if $\Delta(T) \subseteq T \otimes H$. The *coradical filtration* $\mathcal{T} := \{T_n\}_{n \geq 0}$ of a left coideal subalgebra T of H is defined to be

$$T_n := T \cap H_n.$$

For each $n \geq 0$, $\Delta(T_n) \subset H \otimes T_n$, and so the associated graded space

$$\text{gr } T := \bigoplus_{i=0}^{\infty} T(i) \subseteq \text{gr } H; \quad T(i) := ((T \cap H_i) + H_{i-1})/H_{i-1}.$$

satisfies the condition

$$\Delta(\text{gr } T) \subset \text{gr } H \otimes \text{gr } T.$$

Summing up the above we get the following.

Lemma 2.2. *Let H be a connected Hopf k -algebra of finite GK-dimension, and let T be a left coideal subalgebra of H . Then $\text{gr } T$ is a left coideal subalgebra of the commutative graded connected Hopf algebra $\text{gr } H$.*

There is some inconsistency in the literature as to the precise definition of a quantum homogeneous space. For example, Kraehmer in [18] defines a left quantum homogeneous space to be a left coideal subalgebra C of a Hopf algebra H such that H is a faithfully flat left C -module. We adopt in this paper the formally more

restrictive definition used in [23]: a *left quantum homogeneous space* of the Hopf algebra H with a bijective antipode S is a left coideal subalgebra T of H such that H is a faithfully flat left *and* right T -module. In fact, for the connected Hopf algebras which are the object of study in this paper, the distinction is irrelevant, as shown by the result below.

Given a quotient right H -module coalgebra $\pi(H)$ of H , define the *right coinvariants* of π ,

$$H^{\text{co}\pi} := \{h \in H : \sum h_1 \otimes \pi(h_2) = h \otimes \pi(1)\}.$$

Then $H^{\text{co}\pi}$ is a left coideal subalgebra of H . Analogously, the *left coinvariants* of π are

$${}^{\text{co}\pi}H := \{h \in H : \sum \pi(h_1) \otimes h_2 = \pi(1) \otimes h\},$$

a right coideal subalgebra of H .

Proposition 2.3. (Masuoka, [26]) *Let H be a connected Hopf algebra and $T \subseteq H$ a subalgebra.*

- (1) *The antipode S of H is bijective.*
- (2) *T is a left coideal subalgebra if and only if it is a left quantum homogeneous space. In this case, H is a free left and right T -module.*
- (3) *The correspondences $T \mapsto \{\pi_T : H \rightarrow H/T^+H\}$ and $\pi \mapsto H^{\text{co}\pi}$ give a bijective correspondence between the left quantum homogeneous spaces of H and the quotient right H -module coalgebras of H .*
- (4) *There is a parallel version of (3) for right coideal subalgebras of H .*

Proof. (1) This is [26, Proposition 1.2(1)].

(2) Suppose T is a left coideal subalgebra of the connected Hopf algebra H . Since $T_0 = T \cap H_0 = k$, $S(T_0) = T_0$, and so, by [26, Theorem 1.3(1)], H is a left and right faithfully flat T -module. The last sentence is a special case of [26, Proposition 1.4].

(3), (4): Since $S(T_0) = T_0$ as in (2), this is [26, Theorem 1.3(3)] and the version with right and left swapped. \square

Note that, notwithstanding the above result, it is certainly not true that a left coideal subalgebra T of an arbitrary Hopf algebra H is a quantum homogeneous space. For example, let $H = k\langle x \rangle$, the group algebra of the infinite cyclic group. Then $T := k[x]$ is both a left and right coideal subalgebra of H , but *not* a Hopf subalgebra, since $S(T) \not\subseteq T$; and H is *not* a right or left faithfully flat T -module.

The following (presumably well-known) lemma will be used later when classifying quantum homogeneous spaces of small GK-dimension.

Lemma 2.4. *Let T be a left coideal subalgebra of a Hopf algebra H . Suppose $T \not\subseteq H_0$. Then*

$$T_1 := T \cap H_1 \neq T_0.$$

In particular, if H is connected then

$$P(T) := T \cap P(H) \neq k.$$

Proof. Let \widehat{T} be a finite dimensional left subcomodule of T with $\widehat{T} \not\subseteq H_0$; this exists by [28, Theorem 5.1.1]. Let P denote the subcoalgebra of H generated by \widehat{T} ; P is

finite dimensional by [28, Theorem 5.1.1]. Then the finite dimensional algebra P^* acts on P via the right hit action,

$$\leftarrow: P \otimes P^* \rightarrow P : p \otimes f \mapsto p \leftarrow f = \sum f(p_1)p_2.$$

where $\Delta(p) = \sum p_1 \otimes p_2$. Since \widehat{T} is a left subcomodule of T , if $t \in \widehat{T}$ then $\Delta(t) = \sum t_1 \otimes t_2 \in P \otimes \widehat{T}$. Thus $\sum f(t_1)t_2 \in \widehat{T}$ for all $t \in \widehat{T}$, so \widehat{T} is a P^* -submodule of P .

As is well known (see for example [28, §5.2]), the terms $P_i = P \cap H_i$ of the coradical filtration of P are precisely the elements annihilated by the powers $J(P^*)^{i+1}$ of the Jacobson radical $J(P^*)$ of P^* . Now $\widehat{T} \leftarrow J(P^*) \neq 0$ since $\widehat{T} \not\subseteq H_0$. Since P^* is a finite dimensional algebra,

$$\text{Ann}_{\widehat{T}}(J(P^*)) \subsetneq \text{Ann}_{\widehat{T}}(J(P^*)^2),$$

as required. \square

2.3. Quantum homogeneous spaces and the antipode. Recall the following well-known and easy facts, where k can be any field.

Lemma 2.5. *Let H be a Hopf algebra with a bijective antipode.*

- (1) *The map $C \mapsto S(C)$ gives a bijection between the sets of left and right coideal subalgebras of H . This bijection restricts to a bijection between the left and right quantum homogeneous spaces of H .*
- (2) *If C is a left or right coideal subalgebra, then $S(C) = C$ if and only if C is a Hopf subalgebra of H .*

Proof. (1) Let C be a left coideal subalgebra of H and $x \in C$. Then

$$\Delta(S(x)) = \tau \circ (S \otimes S) \circ \Delta(x) = \sum S(x_2) \otimes S(x_1) \in S(C) \otimes H.$$

That is, $S(C)$ is a right coideal subalgebra. Applying S^{-1} shows that the correspondence is bijective. Clearly the correspondence preserves any flatness properties.

- (2) This follows immediately from (1). \square

In the following lemma, we only need to assume that k does not have characteristic 2.

Lemma 2.6. *Let H be a connected Hopf k -algebra.*

- (1) *There exists a k -basis \mathcal{B} of H such that $\mathcal{B} \cap H_n$ spans H_n for all n , and, for all n and all $b \in \mathcal{B} \cap H_n$, there exists $r_b \in H_{n-1}$ such that $S(b) = \pm b + r_b$.*
- (2) *Let $h \in H_n$. There exists $r \in H_{n-1}$ such that $S^2(h) = h + r$.*

Proof. (1) First, S induces the antipode \overline{S} of $\text{gr}H$. Since $\text{gr}H$ is a commutative Hopf algebra by [33, §11.2], \overline{S} has order 2 by [28, Corollary 1.5.12]. Thus the result follows by constructing $\mathcal{B} \cap H_n$ by induction on n , the new elements at stage n being lifts of a basis of \overline{S} -eigenvectors of $H(n)$.

- (2) This follows immediately from (1). \square

Theorem 2.7. *Let H be a connected Hopf k -algebra, with k a field of characteristic 0, and let T be a (left or right) quantum homogeneous space in H .*

- (1) $S^2(T) = T$.
- (2) *Either $(S^2)|_T = \text{id}$ or $|(S^2)|_T| = \infty$.*

Proof. (1) Suppose for definiteness that T is a left coideal subalgebra of H . Under the bijective correspondence of Proposition 2.3(2), $T \leftrightarrow H/T^+H$, and the right coideal subalgebra $S(T)$ corresponds to $H/H(S(T)^+)$. But

$$H(S(T)^+) = HS(T^+) = S(T^+H) = HT^+,$$

where the final equality is Koppinen's lemma, [17, Lemma 3.1]. Thus, applying S now to the pair $\{S(T), HS(T)^+\}$, it follows that the left coideal subalgebra $S^2(T)$ is paired with $H/S(HT^+) = H/T^+H$. By the bijectivity of the correspondence of Proposition 2.3(3), we must have $S^2(T) = T$ as required.

(2) Suppose that $(S^2)|_T \neq \text{id}$, and choose $h \in T_n$ with n minimal such that $S^2(h) \neq h$. By Lemma 2.6(2) there exists $0 \neq r \in H_{n-1}$ such that $S^2(h) = h + r$. By the first part of the theorem, $r \in T$. Hence, by choice of n , $S^2(r) = r$, so that, for all $m \geq 1$,

$$S^{2m}(h) = h + mr.$$

As k has characteristic 0, $S^{2m}(h) \neq h$ for every $m \geq 1$, as required. \square

Note that part (1) of Theorem 2.7 holds, with the same proof, for any *pointed* Hopf algebra H and left coideal subalgebra $T \subset H$ with $S(T_0) = T_0$. This is because Proposition 2.3 is valid in this wider context. But things go wrong with part (2): consider for instance Sweedler's 4-dimensional pointed Hopf algebra [28, Example 1.5.6],

$$K := k\langle g, x, : g^2 = 1, x^2 = 0, xg = -gx \rangle.$$

with $\Delta(g) = g \otimes g$, $\Delta(x) = x \otimes 1 + g \otimes x$, $\epsilon(g) = 1$, $\epsilon(x) = 0$, $S(g) = g$ and $S(x) = -gx$. Here, $|S| = 4$.

In part (2) of the theorem, both alternatives can occur. Of course $S^2 = \text{Id}$ when H is commutative or cocommutative, [28, Corollary 1.5.12]. On the other hand, as we illustrate in §3.5, there are connected Hopf algebras of GK-dimension 3 containing quantum homogeneous spaces T of GK-dimension 2 with $|S^2|_T = \infty$.

3. EXAMPLES OF QUANTUM HOMOGENEOUS SPACES OF CONNECTED HOPF ALGEBRAS

3.1. Coideals in commutative connected Hopf algebras. Recall that k is algebraically closed of characteristic 0, so that H is a commutative affine Hopf k -algebra if and only if $H = \mathcal{O}(G)$ for an affine algebraic k -group G , [38, §1.4].

Theorem 3.1. *Let $H = \mathcal{O}(G)$ as above, with $\dim G = n$. Then the following are equivalent:*

- (1) $H \cong k[X_1, \dots, X_n]$ as algebras;
- (2) G is unipotent;
- (3) H is connected.

Proof. (1) \Rightarrow (2): This is Lazard's theorem, [20].

(2) \Rightarrow (1): [38, Theorem 8.3].

(2) \Leftrightarrow (3): [38, Theorem 8.3]. \square

Now we review the homogeneous G -spaces for G as in the above theorem. $H^{\text{co}\pi}$ and ${}^{\text{co}\pi}H$ have been defined in 2.2

Theorem 3.2. *Let G be an affine unipotent algebraic k -group of dimension n , and let T be a closed subgroup of G with $\dim T = m$. Let $H = \mathcal{O}(G)$ and let I be the defining ideal of T , so I is a Hopf ideal of H with $H/I \cong \mathcal{O}(T)$. Let $\pi : H \rightarrow H/I$ be the canonical epimorphism.*

- (1) *Let $K = H^{\text{co}\pi}$ and $W = {}^{\text{co}\pi}H$. Then K [resp. W] is a left [resp. right] coideal subalgebra of H , with $W = S(K)$ and $K = S(W)$.*
- (2) *K is a Hopf subalgebra of $H \Leftrightarrow K = W \Leftrightarrow T$ is normal in $G \Leftrightarrow K^+H$ is a conormal ideal of H .*
- (3) *K and W are polynomial subalgebras of H in $n - m$ variables, with*

$$H = K[t_1, \dots, t_m] = W[t_1, \dots, t_m]$$

for suitable elements t_i of H , $1 \leq i \leq m$.

- (4) *Every coideal subalgebra of H arises as above. Namely, if K is a left coideal subalgebra of H of dimension $n - m$, then K^+H is a Hopf ideal of H , so $H = \mathcal{O}(T)$ for a closed subgroup T of G with $\dim T = m$, and $K = H^{\text{co}\pi}$ where $\pi : H \rightarrow H/K^+H$.*

Proof. That $K = S(W)$ is a special case of the proof of Theorem 2.7(1). The rest of (1), and (4), are special cases of Proposition 2.3, although of course they were known much earlier.

For part (3), note first that $I = K^+H = W^+H$ by (4), and $H/I \cong k[z, \dots, z_m]$ by Theorem 3.1. That the extensions $K \subset H$ and $W \subset H$ split as stated now follows from basic commutative algebra, taking t_1, \dots, t_m to be any lifts of a set of polynomial generators from H/I to H . The structure of K and W is a theorem of Rosenlicht, [30, Theorem 1], but can also be seen easily by exploiting the basic properties of unipotent groups as in the proof of Theorem 3.3 below.

For (2), see [28, 3.4.2 and 3.4.3]. \square

Theorem 3.3. *Continue with the notation and hypotheses of the previous theorem for H and G .*

- (1) *Let K be a left coideal subalgebra of H , of dimension $n - m$. Then there is a complete flag*

$$(3.1) \quad k = K_0 \subset K_1 \subset \dots \subset K_{n-m} = K \subset \dots \subset K_n = G,$$

where the K_j are left coideal subalgebras of H with $\dim K_j = j$ for all $j = 0, \dots, n$.

- (2) *If K is a Hopf subalgebra of H then the K_j in (3.3) can be chosen to be Hopf subalgebras.*
- (3) *Let each of K and L be a left or right coideal subalgebra of H , with $L \subseteq K$. Then*

$$\text{GKdim } K = \text{GKdim } L \Leftrightarrow K = L.$$

Proof. (1) and (2): By Theorem 3.2(4), there is a closed subgroup T of G , with $\dim T = m$, whose defining ideal is K^+H , and $K = H^{\text{co}\pi}$ where $\pi : H \rightarrow H/K^+H$. The theorem follows by dualising the following standard facts about a unipotent group G in characteristic 0, see for example [16, §17.5]:

- G has a finite chain of normal subgroups $1 = G_0 \subset G_1 \subset \dots \subset G_n = G$ with $\dim G_i = i$ and $G_{i+1}/G_i \cong (k, +)$ for all i ;
- if T is a closed subgroup of G with $T \neq G$ then the normaliser $N_G(T)$ of G is closed, and $\dim N_G(T) > \dim T$.

(3) The construction of T from K , and of the analogous closed subgroup B of G from L , does not depend on whether K and L are right or left coideal subalgebras. In all cases $T \subseteq B$ and (3) follows from the second of the above bullet points. \square

3.2. Cocommutative coideals in connected Hopf algebras. Let C be a cocommutative left quantum homogeneous space in a connected Hopf algebra H . Then cocommutativity ensures that C is in fact a connected sub-bialgebra of H , and so a connected Hopf subalgebra, [28, Lemma 5.2.10]. Thus one can apply the Cartier-Gabrièl-Kostant theorem [28, Theorem 5.6.5], characterising cocommutative connected Hopf algebras over a field of characteristic 0; we find that

$$C \text{ is isomorphic as a Hopf algebra to } \mathcal{U}(\mathfrak{g}),$$

where $\mathfrak{g} = P(C)$ is the Lie algebra of primitive elements of C . Conversely, of course, each subalgebra of the Lie algebra $P(H)$ gives rise to a cocommutative coideal subalgebra of H .

3.3. Commutative coideals in connected Hopf algebras. Let C be a commutative left quantum homogeneous space in a connected Hopf algebra H of finite GK-dimension n . Then C has finite GK-dimension. By Theorem 4.1 below there exists some $m \in \mathbb{Z}_{\geq 0}$ such that $m \leq \text{GKdim} \text{gr} H = n$ and $\text{gr} C \cong k[X_1, \dots, X_m]$. Moreover, $\text{GKdim} C = \text{GKdim} \text{gr} C = m$, with $m = n$ if and only if $C = H$. In particular, C is affine, generated by any choice of lifts of the graded generators X_i to C . Thus C , being commutative, is isomorphic to a factor of the polynomial k -algebra R in m variables. However, proper factors of R have GK-dimension strictly less than m , so $C \cong R$. We have proved:

Proposition 3.4. *Let C be a commutative left quantum homogeneous space in a connected Hopf k -algebra H with $\text{GKdim} H = n < \infty$. Then C is a polynomial algebra in m variables, where $m \leq n$ and $m = n$ if and only if $C = H$.*

3.4. Quantum homogeneous spaces of small Gel'fand-Kirillov dimension. Let C be a left quantum homogeneous space in a connected Hopf k -algebra H , with $\text{GKdim} H < \infty$. We note here that, in dimensions 0 and 1, only classical homogeneous spaces occur in such a Hopf algebra; but a non-classical example occurs already with Gel'fand-Kirillov dimension 2.

(1) Suppose $\text{GKdim}(C) = 0$. Since k is algebraically closed of characteristic 0, H is a domain by [40, Theorem 6.6]. Since C is locally finite dimensional, $C = k$ is the only possibility.

(2) Suppose $\text{GKdim}(C) = 1$. By Lemma 2.4, there exists $0 \neq c \in C \cap P(H)$. Thus C contains the Hopf subalgebra $k[c]$ of H . It is an immediate consequence of Proposition 4.2 below that in fact $k[c] = C$, so that:

the only quantum homogeneous space C with $\text{GKdim} C = 1$ in a connected Hopf algebra H with $\text{GKdim} H < \infty$ is $C = k[c]$ with c primitive.

(3) Suppose $\text{GKdim}(C) = 2$. Then, in addition to the classical examples, namely the enveloping algebras of the two k -Lie algebras of dimension 2 with the usual cocommutative coproduct, one finds that the Jordan plane

$$(3.2) \quad J = k\langle X, Y : [X, Y] = Y^2 \rangle$$

occurs as a coideal subalgebra of a connected Hopf algebra of Gel'fand-Kirillov dimension 3. The details are given in §3.5 below. In particular,

Proposition 3.5. *There exists a quantum homogeneous space of a connected Hopf algebra of finite GK-dimension whose underlying algebra does not admit any structure as Hopf algebra.*

To prove this, one calculates that $\langle [J, J] \rangle = Y^2 J$, so that the abelianisation J^{ab} of J is $k[X, Y]/\langle Y^2 \rangle$. Since, as is easily checked, the abelianisation of a Hopf algebra is always a Hopf algebra, and commutative Hopf algebras in characteristic 0 are semiprime [38, Theorem 11.4], the claim follows. This suggests an obvious project: classify the connected quantum homogeneous spaces of Gel'fand-Kirillov dimension 2. This is listed below as Question 6.2.

3.5. Connected Hopf algebras of small Gel'fand-Kirillov dimension. Let k as usual be algebraically closed of characteristic 0. Zhuang in [40, Examples 7.1, 7.2, Proposition 7.6, Theorem 7.8] classified all the connected Hopf k -algebras H with $\text{GKdim}(H) \leq 3$. In dimension at most 2 one obtains only the enveloping algebras of the Lie algebras \mathfrak{g} with $\dim_k(\mathfrak{g}) \leq 2$, with their standard cocommutative coproducts.

However, for Gel'fand-Kirillov dimension 3, in addition to the cocommutative examples, the classification is completed by two infinite series of (generically) non-commutative, noncocommutative connected Hopf k -algebras $A(\lambda, \mu, \alpha)$ and $B(\lambda)$. We record here the quantum homogeneous spaces of $B(\lambda)$; the classification for the algebras $A(\lambda, \mu, \alpha)$ follows a similar pattern. We shall return to the family $B(\lambda)$ again later to illustrate our results - see Example (5.7).

Let $\lambda \in k$ and define

$$B(\lambda) = k\langle X, Y, Z : [X, Y] = Y, [Z, X] = -Z + \lambda Y, [Z, Y] = \frac{1}{2}Y^2 \rangle.$$

Define $\Delta : B \rightarrow B \otimes B$, $S : B \rightarrow B$ and $\epsilon : B \rightarrow k$ by setting

$$X, Y \in P(B(\lambda)); \quad \Delta(Z) = 1 \otimes Z + X \otimes Y + Z \otimes 1.$$

and

$$\epsilon(X) = \epsilon(Y) = \epsilon(Z) = 0; \quad S(X) = -X, S(Y) = -Y, S(Z) = -Z + XY.$$

One checks routinely that these definitions extend to yield algebra homomorphisms Δ and ϵ and an algebra antihomomorphism S . In [40, Example 7.2] it is proved that (B, Δ, ϵ, S) is a connected Hopf algebra of GK-dimension 3. An easy calculation shows that $S^m(Z) \neq Z$ for any $m > 0$, hence

(3.3) $B(\lambda)$ is a connected Hopf algebra whose antipode has infinite order.

With $W := Z - \frac{1}{2}YX$, a simple calculation shows that, as an algebra, $B(\lambda)$ is isomorphic to the enveloping algebra of the soluble k -Lie algebra \mathfrak{g} with basis X, Y, W and relations

$$[X, Y] = Y, \quad [W, Y] = 0, \quad [X, W] = W - \lambda Y.$$

To fix notation we recall the following:

Definition 3.6. Let R be a ring, σ a ring automorphism of R , and ∂ a σ -derivation of R (that is, an additive map $\partial : R \rightarrow R$ such that $\partial(rs) = \sigma(r)\partial(s) + \partial(r)s$ for all $s, r \in R$). We write $A = R[z; \sigma, \partial]$ and say that A is an *Ore extension* of R provided

- (1) A is a ring, containing R as a subring.
- (2) z is an element of A .
- (3) A is a free left R -module with basis $\{1, z, z^2, \dots\}$.

$$(4) \quad zr = \sigma(r)z + \partial(r) \text{ for all } r \in R.$$

By straightforward calculations, which we leave to the interested reader, one can verify the following:

Proposition 3.7. *The following is a complete list of the non-trivial proper coideal subalgebras of $B(\lambda)$.*

- (1) *Set \mathfrak{g}_α to be the 1-dimensional Lie algebra in $B(\lambda)$ with basis $\{X + \alpha Y\}$ ($\alpha \in k$), and \mathfrak{g}_∞ to be the Lie algebra with basis $\{Y\}$. Then $\{U(\mathfrak{g}_\alpha) : \alpha \in k \cup \{\infty\}\}$ is the complete set of Hopf subalgebras (and also left or right coideal subalgebras) in $B(\lambda)$ of Gel'fand-Kirillov dimension 1.*
- (2) *Let $\delta_\beta, \delta \in \text{Der}_k(U(\mathfrak{g}_\infty))$ be given by*

$$\delta_\beta(Y) = Y + \frac{\beta}{2}Y^2 \text{ and } \delta(Y) = \frac{1}{2}Y^2.$$

Then

$$L_\beta := U(\mathfrak{g}_\infty)[X + \beta Z; \delta_\beta], \text{ for } \beta \in k; \text{ and } L_\infty := U(\mathfrak{g}_\infty)[Z; \delta]$$

are all the left coideal subalgebras of Gel'fand-Kirillov dimension 2 in $B(\lambda)$.

- (3) *The right coideal subalgebras of Gel'fand-Kirillov dimension 2 in $B(\lambda)$ are*

$$R_\beta := U(\mathfrak{g}_\infty)[X + \beta(Z - XY); \delta_{-\beta}], \text{ for } \beta \in k\}$$

and

$$R_\infty := U(\mathfrak{g}_\infty)[Z - XY; -\delta].$$

- (4) *$S(L_\beta) = R_\beta$ for $\beta \in k \cup \{\infty\}$, and vice versa. Moreover, $L_0 = R_0$ is the only Hopf algebra in the lists (2) and (3), and is the only algebra which occurs in both lists.*

4. COIDEAL SUBALGEBRAS OF CONNECTED HOPF ALGEBRAS

4.1. Associated graded algebras and Gel'fand-Kirillov dimension. The starting point for almost everything which follows is:

Lemma 4.1. *Let H be a connected Hopf algebra of finite GK dimension n and $T \subseteq H$ a left coideal subalgebra. Then $\text{gr} T$ is a left coideal subalgebra of $\text{gr} H$, and so there exists some $m \leq n$ such that $\text{gr} T$ forms a graded polynomial algebra in m variables, with $\text{gr} H = \text{gr} T[y_{m+1}, \dots, y_n]$ for some elements y_{m+1}, \dots, y_n of $\text{gr} H$. Moreover, $\text{GKdim } T = \text{GKdim } \text{gr} T = m \in \mathbb{Z}_{\geq 0}$.*

Proof. Most of this is an immediate consequence of Lemma 2.2 and Theorem 3.2(3) and (4). For the final equality of gel'fand-Kirillov dimensions, use [27, Proposition 8.6.5]. \square

The first consequence of Lemma 4.1 is the quantum analogue of the fact (Theorem 3.3(3)) that the normaliser $N_G(T)$ of a closed proper subgroup of a unipotent group in characteristic 0 has dimension strictly greater than that of T . It follows at once from Lemma 4.1 and Theorem 3.3(3).

Proposition 4.2. *Let H be a connected Hopf k -algebra of finite Gel'fand-Kirillov dimension. Let T and S be (left) coideal subalgebras of H such that $S \subseteq T$. Then $S = T$ if and only if $\text{GKdim } S = \text{GKdim } T$.*

4.2. Basic properties of quantum homogeneous spaces of connected Hopf algebras. Next, we show that the excellent homological properties enjoyed by connected Hopf algebras of finite GK-dimension extend to their quantum homogeneous spaces. The proof is a standard application of filtered-graded methods, closely following the case $T = H$ dealt with by Zhuang in [40, Corollary 6.10]. The terminology and notation used in the theorem is standard, and can be found for example in [27], [6] or [15]. In particular, the (*homological*) *grade* of a module M over a k -algebra R is $j_R(M) := \inf\{j : \text{Ext}_R^j(M, R) \neq 0\}$.

Theorem 4.3. *Let H be a connected Hopf k -algebra of finite Gelfand-Kirillov dimension n . Let T be a left coideal subalgebra of H with $\text{GKdim } T = m$.*

- (1) T is a noetherian domain of Krull dimension at most m .
- (2) T is GK-Cohen-Macaulay.
- (3) T is Auslander-regular of global dimension m .
- (4) T is AS-regular of dimension m .

Proof. (1) This follows from [27, Proposition 1.6.6, Theorem 1.6.9 and Lemma 6.5.6].

(2) Let M be a finitely generated (left or right) T -module. Choose a good filtration of M , in the sense of [15, Definition 5.1]. Then $\text{gr } M$ is a finitely generated $\text{gr } T$ -module by [15, Lemma 5.4]. By Lemma 4.1, $\text{gr } T$ is GK-Cohen-Macaulay. In particular,

$$(4.1) \quad j_{\text{gr } T}(\text{gr } M) + \text{GKdim } \text{gr } M = \text{GKdim } \text{gr } T.$$

By Lemma 4.1, $\text{GKdim } T = \text{GKdim } \text{gr } T = m$. Since $\text{gr } T$ is affine and $\text{gr } M$ is a finitely generated $\text{gr } T$ -module, $\text{GKdim } M = \text{GKdim } \text{gr } M$ by [19, Proposition 6.6]. Finally, as in the proof of [3, Theorem 3.9], $j_{\text{gr } T}(\text{gr } M) = j_T(M)$. The result now follows from (4.1).

(3) By Lemma 4.1 and filtered-graded considerations [27, Corollary 7.6.18], T has (right and left) global dimension at most m . Taking M to be the trivial T -module k in (2) yields $j_T(k) = m$. Hence the global dimension of T is m .

The Hopf algebra H is Auslander-Gorenstein by [40, Corollary 6.11], and so, by [23, Proposition 2.3], the left coideal subalgebra T is too. Being Auslander-Gorenstein of finite global dimension, T is by definition Auslander-regular.

(4) By part (2) and the fact that $\text{gldim } T = m$, it remains only to check that the non-zero space $\text{Ext}_T^m(k, T)$ satisfies

$$(4.2) \quad \dim_k \text{Ext}_T^m(k, T) = 1.$$

In the sense of [3, Chapter 2.6], the filtration on the T -module k (coming from the coradical filtration of T) is *good*, [27, 8.6.3]. This yields good filtrations on the T -modules $\text{Ext}_T^j(k, T)$ - let $\text{gr}_*(\text{Ext}_T^j(k, T))$ denote the associated graded $\text{gr } T$ -modules, for $j = 0, \dots, m$. By [3, Proposition 6.10], $\text{gr}_*(\text{Ext}_T^j(k, T))$ is a sub factor of the $\text{gr } T$ -module $\text{Ext}_{\text{gr } T}^j(k, \text{gr } T)$. By Lemma 4.1 $\text{gr } T$ is AS-regular of dimension m , so that $\text{Ext}_{\text{gr } T}^m(k, \text{gr } T) = k$. Thus

$$\dim_k \text{Ext}_T^m(k, T) = \dim_k \text{Ext}_{\text{gr } T}^m(k, \text{gr } T) \leq 1,$$

as required. □

The enveloping algebra $H = U(\mathfrak{g})$ of a finite-dimensional simple complex Lie algebra \mathfrak{g} has Krull dimension $\dim_{\mathbb{C}}(\mathfrak{b})$, where \mathfrak{b} is a Borel subalgebra of \mathfrak{g} , [21]. Thus the inequality in Theorem 4.3(1) is strict in general.

4.3. The Calabi-Yau property for quantum homogeneous spaces. Let A be a k -algebra. For a left $A^e = A \otimes A^{\text{op}}$ -module (that is, A -bimodule) M and k -algebra endomorphisms ν, σ of A , denote by ${}^{\nu}M^{\sigma}$ the A^e -module whose underlying vector space is M , with A^e -action

$$a \cdot m \cdot b = \nu(a)m\sigma(b).$$

for $a, b \in A, m \in M$. If $\nu = \text{Id}$, write M^{σ} rather than ${}^{\text{Id}}M^{\sigma}$.

Definition 4.4. ([8]) An algebra A is ν -twisted Calabi-Yau of dimension d for a k -algebra automorphism ν of A and an integer $d \geq 0$ if

- (1) A is *homologically smooth*, that is, as an A^e -module, A has a finitely generated projective resolution of finite length;
- (2) $\text{Ext}_{A^e}^i(A, A^e) \cong \delta_{i,d}A^{\nu}$ as A^e -modules, where the A^e -module structure on the Ext group is induced by the right A^e -module structure of A^e .

Then ν is uniquely determined up to an inner automorphism and is called the *Nakayama automorphism* of A . Some authors use the term “ ν -skew” rather than “ ν -twisted”. We omit the adjective “twisted” if ν is inner.

Remark 4.5. ([25]) Let A be an AS-Gorenstein k -algebra of dimension d . Taking k to be the *left* A -module annihilated by the augmentation ideal A^+ of A , the one-dimensional space $\text{Ext}_A^d({}_A k, {}_A A)$ is the *left homological integral* of A , denoted \int_A^{ℓ} . From its definition, it has an induced A -bimodule structure: the left A -action is induced by the trivial action on k , whereas the right A -module structure on \int_A^{ℓ} is induced from the right A -module structure on A . Thanks to the AS-Gorenstein hypothesis, this right A -module structure induces a character $\chi : A \rightarrow k$ such that

$$f \cdot a = \chi(a)f.$$

for all $f \in \int_A^{\ell}$ and $a \in A$.

Recall that a Hopf algebra A (with bijective antipode) which is noetherian and AS-Gorenstein of dimension n is twisted Calabi-Yau of dimension n , [8, Proposition 4.5]. In view of [40, Corollary 6.10], the homological corollary of Theorem 2.1, this applies in particular to a connected Hopf k -algebra H of finite GK-dimension n . Moreover, also by [8, Proposition 4.5]), the Nakayama automorphism ν of H is given by

$$\nu = \tau_{\chi}^{\ell} \circ S^2,$$

where S denotes as usual the antipode of H and τ_{χ}^{ℓ} denotes the left winding automorphism of the character $\chi : H \rightarrow k$ defined by \int_H^{ℓ} as in Remark 4.5. That is, $\tau_{\chi}^{\ell}(h) = \sum \chi(h_1)h_2$ for $h \in H$.

Naturally, one asks: does this generalise to a right quantum homogeneous space T of a connected Hopf k -algebra H of finite GK-dimension?

By Theorem 4.3(4) such a right coideal subalgebra T is AS-Gorenstein, and so has a left homological integral \int_T^ℓ , with character $\chi : T \rightarrow k$. *A priori*, the map

$$\tau_\chi^\ell : T \rightarrow H : t \mapsto \sum \chi(t_1)t_2$$

while easily seen to be an algebra homomorphism, might not take values in T . But this *is* in fact so for all such quantum homogeneous spaces T in H , by [23, Lemma 3.9], building on work of [18]. Consequently, we obtain:

Proposition 4.6. *Let H be a connected Hopf k -algebra with finite Gel'fand-Kirillov dimension.*

- (1) *Let T be a right coideal subalgebra of H with $\text{GKdim } T = m$. Then T is twisted Calabi-Yau of dimension m . Retaining the notation introduced above, so in particular χ is the character of the right structure of the left integral of T , the Nakayama automorphism ν of T is*

$$\nu = S^2 \circ \tau_\chi^\ell.$$

- (2) *The same conclusions apply to a left coideal subalgebra T of H , with χ as in (1), with Nakayama automorphism ν of T*

$$\nu = S^{-2} \circ \tau_\chi^r.$$

Proof. (1) That T is homologically smooth follows from [24, Lemma 3.7]. Given that T is AS-regular by Theorem 4.3(4), [24, Theorem 3.6] implies that T is ν -twisted Calabi-Yau, with $\nu = S^2 \circ \tau_\chi^\ell$.

(2) Let T be a left coideal subalgebra of $\mathbf{H} := (H, \mu, \Delta, S, \epsilon)$. By [28, Lemma 1.5.11], $\mathbf{H}' := (H, \mu, \Delta^{op}, S^{-1}, \epsilon)$ is a Hopf algebra, clearly connected, and with the Gel'fand-Kirillov dimensions of H and its subalgebras unchanged, since the algebra structure is the same. However, T is now a *right* coideal subalgebra of \mathbf{H}' , so part (1) can be applied to it. Hence, the Nakayama automorphism is as stated in (2), with a *right* winding automorphism appearing now (with respect to Δ) because the coproduct for \mathbf{H} is Δ^{op} . \square

Here is a typical example to illustrate the proposition.

Example 4.7. Nakayama automorphism of a coideal subalgebra of $B(\lambda)$.

The notation is as introduced in §3.5. So the right coideal subalgebra we consider is $R_\infty := k\langle Y, W \rangle \subseteq B(\lambda)$, where $W := Z - XY$. As noted in Proposition 3.7, $[W, Y] = -\frac{1}{2}Y^2$. Then R_∞ is a right coideal subalgebra of $B(\lambda)$ with $\text{GKdim } R_\infty = 2$, and R_∞ is isomorphic to the Jordan plane. By Proposition 4.6, to compute the Nakayama automorphism of R_∞ we must compute the right R_∞ -module structure of $\int_{R_\infty}^l = \text{Ext}_{R_\infty}^2(k, R_\infty)$.

For an automorphism τ of R_∞ , call $b \in R_\infty$ τ -normal if $\tau(a)b = ba$ for all $a \in R_\infty$. Thus Y is a σ -normal element of R_∞ , where $\sigma(Y) = Y$, $\sigma(W) = W + \frac{1}{2}Y$. Set $\overline{R_\infty} = R_\infty/YR_\infty$. Then $\overline{R_\infty} \cong k[W]$, so that

$$\int_{R_\infty}^l = \text{Ext}_{R_\infty}^1(k, \overline{R_\infty}) \cong {}^1k^1.$$

Hence, by the noncommutative Rees Change of Rings theorem, [8, Lemma 6.6], there is an R_∞ -bimodule isomorphism

$$(4.3) \quad \int_{R_\infty}^l \cong \int_{R_\infty}^l \overset{\sigma^{-1}}{\cong} \cong {}^1k^{\sigma^{-1}} \cong {}^1k^1,$$

where the final isomorphism above holds since twisting by σ^{-1} does not alter the structure of the trivial R_∞ -module. Therefore, by Proposition 4.6,

$$\nu_{R_\infty} = S_{|R_\infty}^2,$$

so that

$$\nu(Y) = Y \text{ and } \nu(W) = W - Y.$$

The above calculation agrees with that carried out for the Jordan plane by other means in, for example [22, §4.2].

Remark 4.8. If T is a Hopf subalgebra of $\mathbf{H} = (H, \Delta, S, \epsilon)$, then both parts of the proposition apply to it. The Nakayama automorphism of a skew Calabi-Yau algebra is unique up to an inner automorphism of the algebra, but in this case H and thus T have no non-trivial inner automorphisms, thanks to the fact that the only units of connected Hopf k -algebras of finite Gel'fand-Kirillov dimension are in k^* , since this is true for their associated graded algebras by Zhuang's Theorem 2.1. Thus, generalising [8, 4.6] for the particular case of connected algebras, we find that, for a Hopf subalgebra T of the connected Hopf k -algebra H of finite Gel'fand-Kirillov dimension,

$$(4.4) \quad S_{|T}^A = \tau_{-\chi}^\ell \circ \tau_\chi^r,$$

where χ is the character of the right structure on \int_T^ℓ . We find this formula rather curious, given that there is no obvious relationship between \int_T^ℓ and \int_H^ℓ .

The following corollary of Proposition 4.6 ought to have a more direct proof.

Corollary 4.9. *Let C be a commutative right or left coideal subalgebra of a connected Hopf k -algebra H of finite Gel'fand-Kirillov dimension. Then $S_C^2 = \text{Id}_C$.*

Proof. Apply Proposition 4.6 to C . Commutativity of C ensures that both the character χ and the automorphism ν are trivial, meaning $\chi = \epsilon$ and $\nu = \text{Id}_C$. Substituting these values in the formula for ν gives the desired conclusion. \square

Note that the corresponding result to the above with ‘‘cocommutative’’ replacing ‘‘commutative’’ is rather trivial, since then C is a cocommutative Hopf subalgebra of H , as was noted in §3.2.

5. INVARIANTS OF QUANTUM HOMOGENEOUS SPACES

5.1. Preliminaries on gradings. Let $A = \bigoplus_{i \geq 0} A(i)$ be a connected \mathbb{N} -graded k -algebra with $\dim_k(A(i)) < \infty$ for all $i \geq 0$. We write

$$h_A(t) = \sum_{i=0}^{\infty} \dim_k A(i) t^i$$

for the *Hilbert series* of A . We need the following well-known and easy lemma.

Lemma 5.1. *Let A, B and C be locally finite connected \mathbb{N} -graded algebras, with $A \cong B \otimes_k C$ as graded algebras. Then*

$$h_A(t) = h_B(t)h_C(t).$$

The next easy lemma is key to the definitions in this section.

Lemma 5.2. *Let R be a commutative connected \mathbb{N} -graded commutative polynomial k -algebra, with homogeneous polynomial generators x_1, \dots, x_n . Let $\mathfrak{m} = \bigoplus_{i>0} R(i) = \langle x_1, \dots, x_n \rangle$ be the graded maximal ideal of R .*

- (1) *Homogeneous elements y_1, \dots, y_n form a set of polynomial generators of R if and only if their images in $\mathfrak{m}/\mathfrak{m}^2$ form a k -basis for this space.*
- (2) *Let C and D be graded polynomial subalgebras of R such that $C \subseteq D$ and $R = C[z_1, \dots, z_t] = D[w_1, \dots, w_r]$; that is, R is a polynomial algebra over C and over D . Then there exist homogeneous elements u_1, \dots, u_n in \mathfrak{m} such that*

$$C = k[u_1, \dots, u_{n-t}], \quad D = k[u_1, \dots, u_{n-r}], \quad R = k[u_1, \dots, u_n].$$

- (3) *The multiset $\sigma(C)$ of degrees of a homogeneous set of polynomial generators of C equals the multiset of degrees of a homogeneous basis of $\mathfrak{m} \cap C / (\mathfrak{m} \cap C)^2$, and hence is independent of the choice of such a generating set.*
- (4) *$\sigma(C) \subseteq \sigma(D)$, with equality if and only if $C = D$. Equivalently, the Hilbert polynomial $h_C(t)$, which equals $\prod_{d \in \sigma(C)} \frac{1}{(1-t^d)}$, divides $h_D(t)$.*

Proof. (1) \Rightarrow : This is trivial.

\Leftarrow : Let y_1, \dots, y_n be homogeneous elements of R whose images modulo \mathfrak{m}^2 form a k -basis for $\mathfrak{m}/\mathfrak{m}^2$. Define $A = k\langle y_1, \dots, y_n \rangle$. Suppose that $A \subsetneq R$, and let s be minimal such that

$$A(s) \subsetneq R(s).$$

Since $R(s) \cap \mathfrak{m}^2$ is spanned by products of pairs of elements in $\{R(i) : i < s\}$,

$$(5.1) \quad R(s) \cap \mathfrak{m}^2 = A(s) \cap \mathfrak{m}^2 \subseteq A(s).$$

Choose $x \in R(s) \setminus A(s)$, so $x \notin \mathfrak{m}^2$. By hypothesis, there exist $\lambda_j \in k$, $1 \leq j \leq n$, such that

$$\hat{x} := x - \sum_j \lambda_j y_j \in \mathfrak{m}^2.$$

Clearly, for all j with $\lambda_j \neq 0$, the degree of y_j is s . But by (5.1) this forces $\hat{x} \in A(s)$, so that $x \in A(s)$, contradicting the choice of x . Therefore we must have $A = R$, and finally considering Krull (or equivalently Gel'fand-Kirillov) dimension shows that y_1, \dots, y_n are polynomial generators of A .

(2) First, note that since $R = C \otimes_k k[z_1, \dots, z_t]$, $\mathfrak{m}^2 \cap C = (\mathfrak{m} \cap C)^2$. Choose homogeneous elements u_1, \dots, u_{n-t} in $\mathfrak{m} \cap C$ whose images modulo $\mathfrak{m}^2 \cap C = (\mathfrak{m} \cap C)^2$ form a k -basis for $\mathfrak{m} \cap C / \mathfrak{m}^2 \cap C$. Thus, by part (1), $C = k[u_1, \dots, u_{n-t}]$. Then $\mathfrak{m} \cap C / \mathfrak{m}^2 \cap C$ embeds in $\mathfrak{m} \cap D / \mathfrak{m}^2 \cap D = \mathfrak{m} \cap D / (\mathfrak{m} \cap D)^2$, so we can extend $\{u_1, \dots, u_{n-t}\}$ to a set of homogeneous elements $\{u_1, \dots, u_{n-r}\}$ of D , of positive degree, whose images modulo $\mathfrak{m}^2 \cap D$ provide a k -basis for $\mathfrak{m} \cap D / (\mathfrak{m} \cap D)^2$. By (1) again, $D = k[u_1, \dots, u_{n-r}]$. A further repeat of the argument extends the set to homogeneous polynomial generators $\{u_1, \dots, u_n\}$ of R .

(3) This is clear from (1) applied to C rather than R , since the degrees and dimensions of the homogeneous components of $\mathfrak{m} \cap C / (\mathfrak{m} \cap C)^2$ are fixed.

(4) This is immediate from (2) and (3). The equivalent formulation in terms of Hilbert series follows from Lemma 5.1. \square

5.2. The signature and the lantern. Lemma 5.2(3) ensures that the following definitions make sense. That the previous parts of the definition apply to T as in part (4) follows from Lemma 4.1.

Definition 5.3. (1) Let R be a connected \mathbb{N} -graded polynomial algebra in n variables, $n < \infty$. The *signature* of R , denoted by $\sigma(R)$, is the ordered n -tuple of degrees of the homogeneous generators.

(2) Let A be an \mathbb{N} -filtered algebra, with filtration $\mathcal{A} = \{A_i\}_{i \geq 0}$, such that the associated graded algebra $\text{gr } A = \bigoplus_i A_i/A_{i-1}$ is a connected \mathbb{N} -graded polynomial algebra in n variables, $n < \infty$. The \mathcal{A} -*signature* of A , denoted by $\sigma_{\mathcal{A}}(A)$, is the signature of $\text{gr } A$ in the sense of (1). Where no confusion is likely refer simply to the *signature* of A , denoted $\sigma(A)$.

(3) With A , \mathcal{A} and $\text{gr } A$ as in (2), suppose that $\text{gr } A$ has m_i homogeneous generators of degree d_i , $1 \leq i \leq t$, with $1 \leq d_1 < \dots < d_t$, so that $\sum_{i=1}^t m_i = n$. Then we write

$$\sigma(A) = (d_1^{(m_1)}, \dots, d_t^{(m_t)}).$$

When $m_i = 1$, the exponent (m_i) is omitted.

(4) Let k be an algebraically closed field of characteristic 0. Let H be a connected Hopf k -algebra of finite Gel'fand-Kirillov-dimension, with a left coideal subalgebra T , with coradical filtration \mathcal{T} as defined in §2.2. Then the *signature* of T , denoted $\sigma(T)$, is the \mathcal{T} -signature of T as defined in (2).

Dualising the above definition, as follows, gives certain benefits, as we shall see. The definition of the *lantern* of a connected Hopf k -algebra, and the key parts (1), (2), (4) and the corollary of Proposition 5.5, are due to Wang, Zhang and Zhuang, [37, Definition 1.2 and Lemma 1.3]. Proofs are given again here for the reader's convenience, in the course of extending their definition.

Definition 5.4. (1) Let $R = \bigoplus_{i=0}^{\infty} R(i)$ denote a connected \mathbb{N} -graded polynomial algebra in n variables, $n < \infty$. Let $\mathcal{D}_R := \bigoplus_{i=0}^{\infty} R(i)^*$ be the *graded dual* of R , so \mathcal{D}_R is a graded cocommutative coalgebra.

(2) The *lantern* $\mathfrak{L}(R)$ of R is the space of primitive elements of \mathcal{D}_R . That is,

$$\mathfrak{L}(R) := P(\mathcal{D}_R),$$

a graded subcoalgebra of \mathcal{D}_R . Note that $\mathfrak{L}(R)$ is non-zero provided $R \neq k$, since then $k = \mathcal{D}_R(0) \subsetneq \mathcal{D}_R$.

(3) Let A be an \mathbb{N} -filtered k -algebra, with filtration $\mathcal{A} = \{A_n\}_{n \geq 0}$, satisfying the hypotheses of Definition 5.3(2). Define the \mathcal{A} -*lantern* of A , denoted by $\mathfrak{L}_{\mathcal{A}}(A)$, to be the lantern of the graded polynomial algebra $\text{gr } A$. Where no confusion is likely, shorten notation to $\mathfrak{L}(A)$.

(4) Let k be an algebraically closed field of characteristic 0. Let H be a connected Hopf k -algebra of finite Gel'fand-Kirillov-dimension, with a left coideal subalgebra T with coradical filtration \mathcal{T} , as in §2.2. Then the *lantern* $\mathfrak{L}(T)$ of T is the \mathcal{T} -lantern of T as defined in (3).

Proposition 5.5. *Let H be a connected Hopf k -algebra of finite Gel'fand-Kirillov dimension n , and let T be a left coideal subalgebra of H , with $\text{GKdim } T = m$. Let $\sigma(H) = (d_1^{(m_1)}, \dots, d_t^{(m_t)})$.*

- (1) The graded dual $\mathcal{D}_H = (\text{gr}H)^*$ is a sub-Hopf algebra of the finite dual $(\text{gr}H)^\circ$; as such, it is isomorphic to the enveloping algebra $U(\mathcal{L}(H))$.
- (2) The lantern $\mathcal{L}(H)$ is a positively graded Lie algebra, with

$$\mathcal{L}(H) = \bigoplus_{i=1}^t \mathcal{L}(H)(d_i), \quad \dim_k \mathcal{L}(H)(d_i) = m_i, \quad \dim_k \mathcal{L}(H) = n.$$

- (3) Let W be the unipotent group of dimension n such that $\text{gr}H \cong \mathcal{O}(W)$. Then $\mathcal{L}(H)$ is the Lie algebra of W .
- (4) $\mathcal{L}(H)$ is generated in degree 1.
- (5) Let Y_T be the closed subgroup of W defined by the Hopf ideal $\text{gr}T^+ \text{gr}H$ of $\text{gr}H$. Let \mathfrak{h}_T be the Lie subalgebra of $\mathcal{L}(H)$ corresponding to Y_T . The graded dual \mathcal{D}_T is a graded left $U(\mathcal{L}(H))$ -module, isomorphic as such to $U(\mathcal{L}(H))/U(\mathcal{L}(H))\mathfrak{h}_T$.
- (6) The lantern $\mathcal{L}(T)$ is the graded quotient $\mathcal{L}(H)/\mathfrak{h}_T$.
- (7) The following are equivalent.
- $\sigma(T) = (e_1^{(r_1)}, \dots, e_s^{(r_s)})$.
 - $\dim_k \mathfrak{L}(T)(e_i) = r_i$ for $i \geq 1$.
 - $h_{\text{gr}T}(t) = \prod_{i=1}^s \frac{1}{(1-t^{e_i})^{r_i}}$.

Proof. (1) Let $\mathfrak{m} = \bigoplus_{i \geq 1} H(i)$ be the graded maximal ideal of $\text{gr}H$. By construction,

$$\mathcal{D}_H = \{f \in (\text{gr}H)^\circ : f(\mathfrak{m}^j) = 0 \text{ for } j \gg 0\}.$$

By [28, Proposition 9.2.5], \mathcal{D}_H is a sub-Hopf algebra of $(\text{gr}H)^\circ$; more precisely,

$$\mathcal{D}_H = U(P(\mathcal{D}_H)),$$

the enveloping algebra of the Lie algebra of primitive elements of \mathcal{D}_H . Note also that, by Definition 5.4(1) and [28, Lemma 9.2.4]

$$(5.2) \quad \mathcal{L}(H) = P(\mathcal{D}_H) = \{f \in \mathcal{D}_H : f(\mathfrak{m}^2 + k1) = 0\}.$$

(2) Since \mathcal{D}_H is a graded Hopf algebra, its subspace $\mathcal{L}(H)$ of primitive elements is a graded Lie algebra: for $d \geq 1$,

$$\mathcal{L}(H)(d) = \{f \in (H(d))^* : f(\mathfrak{m}^2 \cap H(d)) = 0\}.$$

Thus, for all $d \geq 1$,

$$(5.3) \quad \dim_k \mathcal{L}(H)(d) = \dim_k (H(d)/\mathfrak{m}^2 \cap H(d)) = \dim_k ((\mathfrak{m}/\mathfrak{m}^2)(d)).$$

By Lemma 5.2(3), this completes the proof of (2).

(3) This follows from (5.2) and the definition of the Lie algebra of an algebraic group, [16, §9.1].

(4) This is [2, Lemma 5.5].

(5) Appealing again to [16, §9.1] and noting (5.2),

$$(5.4) \quad \begin{aligned} \mathfrak{h}_T &= \{f \in \mathcal{L}(H) : f((\text{gr}T)^+ \text{gr}H) = 0\} \\ &= \{f \in \mathcal{D}_H : f((\text{gr}T)^+ \text{gr}H + \mathfrak{m}^2 + k1) = 0\}. \end{aligned}$$

The convolution product makes \mathcal{D}_T into a left module over $\mathcal{D}_H = U(\mathcal{L}(H))$: namely, for $u \in U(\mathcal{L}(H))$, $f \in \mathcal{D}_T$ and $t \in \text{gr}T$,

$$uf(t) = u(t_1)f(t_2).$$

It is clear that, with respect to this action, the restriction map

$$\rho : U(\mathcal{L}(H)) = (\text{gr}H)^* \longrightarrow \mathcal{D}_T = (\text{gr}T)^*$$

is a homomorphism of left $U(\mathcal{L}(H))$ -modules. From (5.4), $\eta_T \subseteq \ker \rho$, and hence

$$(5.5) \quad U(\mathcal{L}(H))\eta_T \subseteq \ker \rho.$$

It remains to check that equality holds in (5.5). Take a graded basis $\{u_1, \dots, u_n\}$ of $\mathcal{L}(H)$, where $\{u_1, \dots, u_m\}$ is a dual basis to a set x_1, \dots, x_m of graded polynomial generators of $\text{gr}T$ and where $\{u_{m+1}, \dots, u_n\}$ forms a basis of η_T . Let $x_i \in H(d_i)$, so $u_i \in H(d_i)^*$ for $i = 1, \dots, n$. Thus $U(\mathcal{L}(H)) = \mathcal{D}_H$ is graded, where, for $j \geq 0$, a basis of $U(\mathcal{L}(H))(j)$ is given by the ordered monomials $u_1^{r_1} \dots u_n^{r_n}$ for which $\sum_{i=1}^n r_i d_i = j$. Then $U(\mathcal{L}(H))\eta_T(j)$ is spanned by those ordered monomials in the u_i of degree j for which $r_i > 0$ for some $i > m$. Comparing the dimensions of $\mathcal{D}_T(j)$ with $(U(\mathcal{L}(H))/U(\mathcal{L}(H)\eta_T))(j)$ for $j \geq 0$ now yields equality in (5.5).

(6) It follows from (5.3) and (5.4) that

$$(5.6) \quad \begin{aligned} (\mathcal{L}(H)/\eta)(d) &\cong \mathcal{L}(H)(d)/\eta(d) \\ &\cong \{f \in (H(d))^* : f((\mathfrak{m}^2 + (\text{gr}T)^+ \text{gr}H) \cap H(d)) = 0\} \\ &\cong ((\text{gr}T)^+ / (\mathfrak{m}^2 \cap (\text{gr}T)^+)(d))^*. \end{aligned}$$

Since the final term above is $\mathcal{L}(T)(d)$, this proves (6).

(7) The equivalence of (a) and (c) was noted in Lemma 5.2(4) and its proof. The equivalence of (a) and (b) is (5.3). \square

In the literature, a finitely generated positively graded (and hence nilpotent) Lie algebra which is generated in degree 1 is called a *Carnot Lie algebra*; they are important in a number of branches of mathematics, for example in Riemannian geometry. For a brief review with references, one can consult for instance [9]

The *classical* part of the picture described by the theorem, that is the *connected cocommutative* aspect familiar from basic Lie theory, is as follows.

Corollary 5.6. *Retain the notation of the above theorem. Then the following are equivalent:*

- (1) $\mathcal{L}(H)$ is abelian;
- (2) $\mathcal{L}(H) = \mathcal{L}(H)(1)$;
- (3) H is cocommutative;
- (4) $H \cong U(\mathfrak{g})$ as a Hopf algebra, for some n -dimensional Lie algebra \mathfrak{g} ;
- (5) $\text{gr}H$ is cocommutative;
- (6) W is abelian, $W \cong (k, +)^n$;
- (7) $\sigma(H) = (1^{(n)})$.

Proof. (1) \Rightarrow (2): If $\mathcal{L}(H)$ is abelian, then $\mathcal{L}(H) = \mathcal{L}(H)(1)$ by Proposition 5.5(4).

(3) \Leftrightarrow (4) This is a special case of the Cartier-Kostant-Gabri el theorem [28, Theorem 5.6.5].

(5) \Leftrightarrow (6): That W is abelian if and only if its coordinate ring $\text{gr}H$ is cocommutative is immediate by duality. That, in this case, $W \cong (k, +)^n$ is a consequence of the structure of abelian algebraic groups in characteristic 0, [16, Corollary 17.5, Exercises 15.11, 17.7].

(2) \Leftrightarrow (7): This is a special case of Proposition 5.5.

(7) \Rightarrow (4): Suppose $\sigma(H) = (1^{(n)})$. Then $\text{gr}H$ is generated by elements of $H(1)$, that is, by primitive elements. Hence $\text{gr}H$ is cocommutative.

(3) \Rightarrow (5): Trivial.

(5) \Rightarrow (7): If $\text{gr}H$ is cocommutative, then by the Cartier-Kostant-Gabriël theorem [28, Theorem 5.6.5] it is an enveloping algebra as a Hopf algebra, so generated by the space $H(1)$ of primitive elements. That is, $\sigma(H) = (1^{(n)})$. \square

We return to the family of examples $B(\lambda)$ to illustrate aspects of Proposition 5.5.

Example 5.7. Signature and lantern of $B(\lambda)$. The connected Hopf algebras $B(\lambda)$, for $\lambda \in k$, were recalled from [40] in §3.5. They constitute one of two infinite families of connected Hopf k -algebras of Gel'fand-Kirillov dimension 3. Starting from the description of the coideal subalgebras of $B(\lambda)$ in Proposition 3.7, one easily calculates the following facts:

- (1) $\sigma(B(\lambda)) = (1^{(2)}, 2)$.
- (2) $\mathcal{L}(B(\lambda))$ is the Heisenberg Lie algebra of dimension 3; equivalently, the group W with coordinate ring $\text{gr}B(\lambda)$ is the 3-dimensional Heisenberg group. Here, $\text{gr}B(\lambda) = k[X, Y, Z]$ in the obvious “lazy” notation for lifts of elements to the associated graded algebra.
- (3) The unique two-dimensional Hopf subalgebra of $B(\lambda)$, namely $k\langle X, Y : [X, Y] = Y \rangle$, labelled $L_0 = R_0$ in Proposition 3.7, has signature $(1^{(2)})$, with $\text{gr}L_0 = k[X, Y]$.
- (4) The remaining two-dimensional left and right coideal subalgebras L_β, R_β , for $\beta \in \{\infty\} \cup (k \setminus \{0\})$, have signature $(1, 2)$. For all such β , $\text{gr}L_\beta = k[Y, Z]$ and $\text{gr}R_\beta = k[Y, Z - XY]$.

5.3. Numerology. The first of the two results in this subsection assembles what we know about the signature of a connected Hopf algebra H . The second theorem gives a parallel account of the known numerical constraints on the signature of a quantum homogeneous space of such an H . For convenience, some results obtained earlier in the paper are restated here.

Theorem 5.8. *Let k be an algebraically closed field of characteristic 0, and let H be a connected Hopf k -algebra of finite Gel'fand-Kirillov dimension n . Let*

$$\sigma(H) = (d_1^{(m_1)}, \dots, d_u^{(m_u)}), \text{ so } h_{\text{gr}H}(t) = \prod_{i=1}^u \frac{1}{(1 - t^{d_i})^{m_i}}.$$

- (1) (Wang, Zhang, Zhuang, [37, Lemma 1.3(d)]) *If $n > 1$, then $m_1 \geq 2$. That is, $\dim_k P(H) \geq 2$ if $\text{GKdim} H \geq 2$.*
- (2) (NO GAPS) $\{d_1, \dots, d_t\} = \{1, \dots, t\}$.
- (3) For all $i = 1, \dots, t$,

$$m_i \leq \frac{1}{i} \sum_{d|i} \mu(d) m_1^{(i/d)},$$

where $\mu : \mathbb{N} \rightarrow \mathbb{N}$ is the Mobius function.

Proof. (1) Since $\mathcal{L}(H)$ is generated in degree 1 by Proposition 5.5(4), $\mathcal{L}(H)$ and hence H would be one-dimensional if $\dim_k \mathcal{L}(H) = 1$.

(2) A lemma on \mathbb{N} -graded Lie algebras which are generated in degree 1, proved easily by induction, implies that, for all $i \geq 1$,

$$\mathcal{L}(H)(i+1) = [\mathcal{L}(H)(1), \mathcal{L}(H)(i)].$$

Thus (2) follows from this and Proposition 5.5(2).

(3) Let $1 \leq i \leq t$. By (2) and Proposition 5.5(7), $m_i = \dim_k \mathcal{L}(H)(i)$, so the bound follows Proposition 5.5(4) and from the well-known Witt formula for the dimension of the i th graded summand of the free Lie algebra on m_1 generators of degree 1, [39], [32]. \square

Theorem 5.9. *Let k and H be as in Theorem 5.8. Let each of K and L be right or left coideal subalgebras of H , of Gel'fand-Kirillov dimensions m and ℓ respectively, with $L \subseteq K$. Let $\sigma(K) = (e_1^{(r_1)}, \dots, e_s^{(r_s)})$ and $\sigma(L) = (f_1^{(q_1)}, \dots, f_p^{(q_p)})$.*

- (1) $m = \sum_i r_i \geq \sum_j q_j = \ell$.
- (2) $\ell = m$ if and only if $L = K$.
- (3) $\sigma(L)$ is a sub-multiset of $\sigma(K)$. That is,

$$h_{\text{gr}L}(t) | h_{\text{gr}K}(t).$$

Equality holds (of multisets and of Hilbert polynomials) if and only if $L = K$.

- (4) If $K \neq k$ then $e_1 = 1$. Similarly, of course, for L .

Proof. (1) Immediate from Lemma 4.1 and the definition of the signature, Definition 5.3(4).

(2) This is a consequence of the corresponding result when H is commutative, Theorem 3.3(4), together with Lemma 4.1.

(3) Immediate from the definition and from Lemma 5.2(4).

(4) This is Lemma 2.4. \square

Remarks 5.10. (1) One might expect that Theorem 5.8(1) applies more generally, namely to quantum homogeneous spaces rather than just to Hopf algebras, especially in the light of Theorem 5.9(4). But this is not the case, as is illustrated by $H = B(\lambda)$, see Example 5.7(4).

(2) Similarly, the No-Gaps Theorem 5.8(3) does *not* extend to quantum homogeneous spaces. This is shown by the example below.

Example 5.11. A quantum homogeneous space with signature $(1^{(2)}, 3)$.

The example is one of the families listed in the classification of connected Hopf algebras of Gel'fand-Kirillov dimension 4, [37, Example 4.5]. Let $a, b, \lambda_1, \lambda_2 \in k$, and let E be the k -algebra generated by X, Y, Z, W , subject to the following relations:

$$\begin{aligned} [Y, X] &= [Z, Y] = 0, [Z, X] = X, [W, X] = aX, \\ [W, Y] &= bX, [W, Z] = aZ - W + \lambda_1 X + \lambda_2 Y. \end{aligned}$$

It is shown in [37] that there is a Hopf algebra structure on E such that E is connected with $\text{GKdim} E = 4$. Namely, one defines $X, Y, Z, W \in \ker \epsilon$, and $\Delta : E \rightarrow E \otimes E$ is fixed by setting $X, Y \in P(E)$ and

$$\begin{aligned} \Delta(Z) &= 1 \otimes Z + X \otimes Y - Y \otimes X + Z \otimes 1, \\ \Delta(W) &= 1 \otimes W + W \otimes 1 + Z \otimes X - X \otimes Z + X \otimes XY + XY \otimes X. \end{aligned}$$

In [37, Proposition 4.8] it is shown that

$$\sigma(E) = (1^{(2)}, 2, 3);$$

indeed one can see from the definition of Δ that $Z \in E_2 \setminus E_1$ and $W \in E_3 \setminus E_2$. Define $T = k\langle X, Y, W - XZ \rangle$ and confirm easily that

$$T = k[X, Y][(W - XZ); \partial],$$

where $\partial(X) = aX - X^2$ and $\partial(Y) = bX$. Note that $Z \notin T$ and that

$$\begin{aligned} \Delta(W - XZ) &= 1 \otimes (W - XZ) + (W - XZ) \otimes 1 \\ &\quad + 2(XY \otimes X) + 2(X \otimes Z) + X^2 \otimes Y - Y \otimes X^2 \\ &\in T \otimes E. \end{aligned}$$

Thus T is a right coideal subalgebra of E , with

$$\sigma(T) = (1^{(2)}, 3).$$

6. QUESTIONS AND DISCUSSION

Some questions concerning connected Hopf algebras are listed in the survey article [4]. We don't repeat those questions here, focussing instead on the possible role of quantum homogeneous spaces in the study of these Hopf algebras. As elsewhere in this paper, k is algebraically closed of characteristic 0.

6.1. Classification of algebras.

Question 6.1. Let T be a k -algebra with a finite dimensional filtration $\mathcal{F} = \{T_i : i \geq 0\}$ with $T_0 = k$, such that $\text{gr}_{\mathcal{F}}T$ is a commutative polynomial algebra in finitely many variables. What conditions on T and \mathcal{F} are required to ensure that T is a quantum homogeneous space of a connected Hopf algebra with $\text{GKdim}H < \infty$, with \mathcal{F} the coradical filtration of T ?

This question appears to be difficult. There are some easy observations to be made: some numerical constraints on T can be read off from Theorem 5.9; and T has to admit a 1-dimensional representation, the restriction of the counit. If the additional hypothesis is imposed, that $\text{gr}_{\mathcal{F}}T$ is generated by finitely many elements of degree 1, then T_1 is a Lie algebra with respect to the bracket $[a, b] = ab - ba$. Hence, by the universal property of enveloping algebras, T is an epimorphic image of $U(T_1)$ (assuming only $\text{gr}T$ affine commutative, not necessarily a polynomial algebra). This observation is due to Duflo, [11], [27, Proposition 8.4.3]. Such algebras are called *almost commutative* in [27]. But the Jordan plane is a quantum homogeneous space of a connected Hopf algebra, by Proposition 3.7. Since the Jordan plane is shown in [27, Proposition 14.3.9] *not* to be almost commutative, it follows that the class of k -algebras defined by Question 6.1 is not a subclass of the class of almost commutative algebras.

One can refine Question 6.1 in various ways. For example, one can ask what is needed so that T is a connected Hopf algebra, not just a quantum homogeneous space. In particular, we asked in [4, Question L] whether every such T is the enveloping algebra of a finite dimensional Lie algebra. The answer to this is "No", as we shall demonstrate in [5].

In a second direction, one can restrict the size of T in Question 6.1 and ask for a classification. As noted in 3.4, if $\text{GKdim}T \leq 1$, then T is either k or $k[x]$, with x primitive. Beyond dimension 1, the question is open:

Question 6.2. What are the quantum homogeneous spaces T with $\text{GKdim}T = 2$ in connected Hopf k -algebras H of finite Gel'fand-Kirillov dimension?

If this is too easy, one can consider the same question for H a Hopf domain of finite Gel'fand-Kirillov dimension, not necessarily connected. Note that the corresponding question for $T = H$ of dimension 2 has been answered in [13].

6.2. Classification via the signature. The signature may offer a useful way to organise the possible H and the possible T . So, for example, in the sense that all finite dimensional Lie k -algebras are “known”, we can describe all the connected Hopf k -algebras with H with $\sigma(H) = (1^{(n)})$ - namely, they are the enveloping algebras $H = U(\mathfrak{g})$ with $\mathfrak{g} = P(H)$. And the same holds for the quantum homogeneous spaces T with this signature, by §3.2. Next up might be the *primitively thick* connected Hopf algebras H , these being the ones with signature $\sigma(H) = (1^{(n-1)}, 2)$. These are completely described, along with their coideal subalgebras, in [5]. There are various options as to what to consider next; for example, one can aim to classify the *primitively thin* algebras, those at the opposite extreme from the thick ones. Namely:

Question 6.3. Classify the connected Hopf k -algebras H with $\sigma(H) = (1^{(2)}, \dots)$. Do the same for the quantum homogeneous spaces T with $\sigma(T) = (1, \dots)$.

Note that the second part of Question 6.3 incorporates Question 6.2.

6.3. Quantisations of unipotent groups. Recall that if H is connected Hopf k -algebra of finite Gel'fand-Kirillov dimension, then $\text{gr}H$ is the coordinate ring of a unipotent k -group U . Naturally, one should consider reversing this passage to the associated graded algebra of H :

Question 6.4. Which unipotent k -groups U admit non-trivial “lifts”? That is, for which such U does there exist a *noncommutative* connected Hopf algebra H with $\text{gr}H \cong \mathcal{O}(U)$?

Of course the answer to Question 6.4 is known when U is abelian: if $U \cong (k, +)^n$ for any $n > 1$, one can take $H = U(\mathfrak{g})$ where \mathfrak{g} is any non-abelian Lie algebra of dimension n . So perhaps a first step on the route towards answering Question 6.4 might be:

Question 6.5. Is there a unipotent group U with $\dim U > 1$ and with $\text{Lie}U$ Carnot, which has no non-trivial lift?

6.4. Complete flags of quantum homogeneous spaces. If H is a *commutative* connected Hopf k -algebra of finite Gel'fand-Kirillov dimension n , then, as we noted in Theorem 3.3, there is chain of $n+1$ coideal subalgebras (in fact Hopf subalgebras) from k to H . The same is not always true in the *cocommutative* case, where (by an easy argument making use of [7, §3.1, Examples (iii),(iv)]), such a flag exists if and only if \mathfrak{g} has solvable radical \mathfrak{r} with $\mathfrak{g}/\mathfrak{r}$ isomorphic to a finite direct sum of copies of $\mathfrak{sl}(2, k)$. Bearing these cases in mind and aiming to develop a generalisation of the solvable radical in the connected Hopf setting, one might propose the following:

Question 6.6. (i) Is there a good structure theory for connected Hopf k -algebras H which possess a complete flag of coideal subalgebras K_i ,

$$k = K_0 \subset K_1 \subset \dots \subset K_n = H,$$

with K_i^+H a Hopf ideal of H ?

(ii) If H is *any* connected Hopf k -algebra of finite Gel'fand-Kirillov dimension, does H have a maximal $\text{ad}H$ -normal Hopf subalgebra R with the property (i)?

- (iii) Given (ii), what can be said about the Hopf algebra H/R^+H ?
 (iv) How does the class of Hopf algebras in (i) compare with the IHOEs studied in [7]?

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