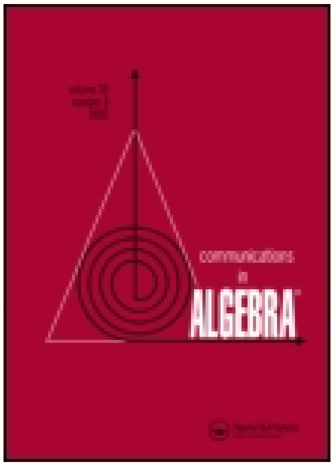

Communications in Algebra
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/lagb20

A Lie-Rinehart Algebra with No Antipode

Ulrich Krähmer [a] & Ana Rovi [a]
[a] School of Mathematics and Statistics, University of Glasgow, Glasgow, UK
Published online: 06 Jul 2015.

To cite this article: Ulrich Krähmer & Ana Rovi (2015) A Lie-Rinehart Algebra with No Antipode, Communications in Algebra, 43:10, 4049-4053, DOI: 10.1080/00927872.2014.896375

To link to this article: http://dx.doi.org/10.1080/00927872.2014.896375

Communications in Algebra®, 43: 4049–4053, 2015
Published with license by Taylor & Francis
ISSN: 0092-7872 print/1532-4125 online
DOI: 10.1080/00927872.2014.896375

A LIE–RINEHART ALGEBRA WITH NO ANTIPODE

Ulrich Krähmer and Ana Rovi
School of Mathematics and Statistics, University of Glasgow, Glasgow, UK

The aim of this note is to communicate a simple example of a Lie–Rinehart algebra whose enveloping algebra is not a Hopf algebroid, neither in the sense of Böhm and Szlachányi, nor in the sense of Lu.

Key Words: Hopf algebroids; Lie algebroids; Lie–Rinehart algebras.

2010 Mathematics Subject Classification: 16T05; 57T05; 16S30.

1. INTRODUCTION

The enveloping algebra of a Lie algebra is a classical example of a Hopf algebra. Hence it is natural to ask whether the enveloping algebra of a Lie algebroid [12] or more generally of a Lie–Rinehart algebra [13] carries the structure of a Hopf algebroid. It turns out that they always are *left bialgebroids* (introduced under the name $\times_R$-bialgebras by Takeuchi [15]), see [16], and in fact *left Hopf algebroids* (introduced under the name $\times_R$-Hopf algebras by Schauenburg [14]), see [6, Example 2]; see also [5, 11].

However, there is also the definition of a Hopf algebroid due to Lu [9] and the one due to Böhm and Szlachányi [2] (the latter will be called *full Hopf algebroids* from now on), which both assume the existence of an antipode satisfying certain axioms. The aim of the present paper is to communicate a concrete example of a Lie–Rinehart algebra whose universal enveloping algebra is not a Hopf algebroid in either of these two settings.

This clarifies further the relation between the three concepts: it is well-known and easily seen that every full Hopf algebroid is a left Hopf algebroid, see [2]. In [1], an example of a full Hopf algebroid was given that is not a Hopf algebroid in the sense of [9], but to the best of our knowledge, it is not known whether every Hopf algebroid in the sense of Lu satisfies the axioms of a full or at least of a left Hopf algebroid, and whether every left Hopf algebroid admits an antipode satisfying

Received September 23, 2013; Revised February 11, 2014. Communicated by T. Lenagan.
Address correspondence to Ulrich Krähmer, School of Mathematics and Statistics, University of Glasgow, Glasgow G12 8QW, UK; E-mail: ulrich.kraehmer@glasgow.ac.uk

either of the definitions of [2, 9] (a counterexample announced in [8, Remark 3.12] did not appear in print).

In the light of [8, Proposition 3.11], it is known that the enveloping algebras of Lie algebroids [3] and of the Lie–Rinehart algebras associated to Poisson algebras [4, Section (3.2)] are full Hopf algebroids. However, here we prove the following theorem.

Theorem 1.1. *Let K be a field, $R := K[x, y]/\langle x \cdot y, x^2, y^2\rangle$, L be the 1-dimensional Lie algebra with basis $\{\alpha\}$, and $E \in \mathrm{Der}_K(R)$ be the derivation with $E(x) = y$, $E(y) = 0$.*

1. *There is a Lie–Rinehart algebra structure on (R, L) with R-module structure on L given by $x \cdot \alpha = y \cdot \alpha = 0$ and anchor map given by $\rho(\alpha) = E$.*
2. *There is no right $V(R, L)$-module structure on R that extends multiplication in R.*
3. *$V(R, L)$ is neither a full Hopf algebroid, nor a Hopf algebroid in the sense of* [9].

The note is structured as follows. In Section 2, we recall some basic definitions and prove the implication 2. $\Rightarrow$ 3. of Theorem 1.1. In Section 3, we provide a construction method of Lie–Rinehart algebras whose enveloping algebras satisfy part 2, for which the Lie–Rinehart algebra from 1 is a basic example.

2. BACKGROUND

This section contains background on Lie–Rinehart algebras [13], see also [4, 7, 8, 11] for more information. For the corresponding differential geometric notion of a Lie algebroid see [12] and for example [10] for further details.

We fix a field K. An unadorned $\otimes$ denotes the tensor product of K-vector spaces.

Definition 2.1. A Lie–Rinehart algebra consists of the following elements:

1. A commutative K-algebra $(R, \cdot)$;
2. A Lie algebra $(L, [-, -]_L)$ over K;
3. A left R-module structure $R \otimes L \to L$, $r \otimes \xi \mapsto r \cdot \xi$, $r \in R$, $\xi \in L$; and
4. An R-linear Lie algebra homomorphism $\rho : L \to \mathrm{Der}_K(R)$ satisfying

$$[\xi, r \cdot \zeta]_L = r \cdot [\xi, \zeta]_L + \rho(\xi)(r) \cdot \zeta, \quad r \in R, \xi, \zeta \in L. \tag{2.1}$$

The map ρ is referred to as the anchor map.

There are two fundamental examples: if R is any commutative algebra, one can take L to be $\mathrm{Der}_K(R)$ with its usual Lie algebra and R-module structure, and $\rho = \mathrm{id}$. The other extreme is $R = K$ and $\rho = 0$, L being any Lie algebra.

In his paper [13], Rinehart generalised the construction of the universal enveloping algebra of a Lie algebra to Lie–Rinehart algebras, see Section 2 therein for the precise construction. The result is an associative K-algebra $V(R, L)$ that is generated by the (sum of the) images of a K-algebra map

$$R \longrightarrow V(R, L)$$

and a Lie algebra map

$$(L, [-, -]_L) \longrightarrow (V(R, L), [-, -]), \quad \xi \longmapsto \bar{\xi},$$

where $[-, -]$ denotes the commutator in $V(R, L)$. As Rinehart, we do not distinguish between an element in R and its image in $V(R, L)$ which is justified as the first map is always injective. The construction is such that in $V(R, L)$ one has for all $r \in R$, $\xi \in L$

$$[\bar{\xi}, r] = \rho(\xi)(r), \quad r\bar{\xi} = \overline{r \cdot \xi}, \tag{2.2}$$

where the product in $V(R, L)$ is denoted by concatenation.

As indicated in the introduction, $V(R, L)$ has the structure of a left Hopf algebroid. Its counit endows R with the structure of a left $V(R, L)$-module, in such a way that the induced action of $r \in R$ is given by multiplication, and the induced action of $\xi \in L$ is given by the anchor map. The following fact is well known and yields the implication 2. ⇒ 3. in Theorem 1.1.

Lemma 2.2. *If H is either a full Hopf algebroid or a Hopf algebroid in the sense of Lu, with antipode $S : H \to H$, left counit $\varepsilon : H \to R$, and source and target maps $s, t : R \to H$, then defining for $h \in H, r \in R$*

$$rh := \varepsilon(S(h)s(r)) \tag{2.3}$$

yields a right H-module structure on R for which the underlying left R-action on R is given by left multiplication.

Proof. The canonical left action of a left bialgebroid H on the base algebra R is given by $hr := \varepsilon(hs(r)) = \varepsilon(ht(r))$, and the antipode of a Hopf algebroid is an algebra antihomomorphism (see [1, Proposition 4.4] respectively [9, Definition 4.1] for the two different notions). Hence (2.3) defines a right action of H on R. Finally, one has $S \circ t = s$ (see [1, Definition 4.1 (iii)] respectively [9, Definition 4.1.2.]), so $rt(q) = \varepsilon(S(t(q))s(r)) = \varepsilon(s(q)s(r)) = qr$ for all $q, r \in R$. □

3. A LIE–RINEHART ALGEBRA WITHOUT FLAT RIGHT CONNECTION ON *R*

We now prove Theorem 1.1, 1. and 2. We begin by considering more generally Lie–Rinehart algebras (R, L) whose R-module structure on L is given by a character $\chi : R \to K$.

Lemma 3.1. *Let $(R, \cdot)$ be a commutative K-algebra, $(L, [-, -]_L)$ be a Lie algebra, and $\rho : L \to \mathrm{Der}_K(R)$ be a Lie algebra map. Define an R-module structure on L by $r \cdot \xi := \chi(r)\xi$, where $\chi : R \to K$ is a character on R. Then (R, L) is a Lie–Rinehart algebra if and only if ρ is R-linear and $\rho(\xi)(r) \in \ker \chi$ for all $r \in R, \xi \in L$.*

Proof. This follows as the Leibniz rule (2.1) takes the form

$$[\xi, \chi(r)\zeta]_L = \chi(r)[\xi, \zeta]_L + \chi(\rho(\xi)(r))\zeta$$

and hence by the K-linearity of the bracket becomes equivalent to $\rho(\xi)(r) \in \ker\chi$. □

Note that for these examples, $[-.-]_L$ is even R-linear, so L is a Lie algebra over R. However, in general we have $\rho \neq 0$.

Assume now that (R, L) is a Lie–Rinehart algebra as in the above lemma, and that multiplication in R can be extended to a right $V(R, L)$-module structure on R. Denote by $\partial(\xi) \in R$ the element obtained by acting with $\xi \in L$ on $1 \in R$ under this right action. This defines a K-linear map $\partial : L \to R$, and in $V(R, L)$ we have

$$\rho(\xi)(r) = [\bar{\xi}, r] = \bar{\xi}r - r\bar{\xi} = \bar{\xi}r - \overline{r \cdot \xi} = \bar{\xi}r - \chi(r)\bar{\xi},$$

so by acting with this element on $1 \in R$, one sees that this map ∂ satisfies

$$\rho(\xi)(r) = \partial(\xi) \cdot (r - \chi(r)). \tag{3.1}$$

A K-linear map ∂ with this property defines a right $V(R, L)$-module structure extending multiplication on R if and only if it satisfies the condition $\partial([\xi, \zeta]_L) = \rho(\xi)(\partial(\zeta)) - \rho(\zeta)(\partial(\xi))$. It also corresponds to a generator of the Gerstenhaber bracket on $\Lambda_R L$, see [4], but we shall not need these facts.

Proof of Theorem 1.1, 1. and 2. The first part is verified by explicit computation; the Lie–Rinehart algebra is of the form as in Lemma 3.1 with χ given by $\chi(x) = \chi(y) = 0$.

For 2., take $r = x$ and $\xi = \alpha$ in (3.1). One obtains $y = E(x) = \rho(\alpha)(x) = \partial(\alpha) \cdot x$. However, there is no element $z \in R$ such that $y = z \cdot x$. $\square$

Carrying out Rinehart's construction explicitly yields a presentation of the associative K-algebra $V(R, L)$ in terms of generators $x, y, \bar{\alpha}$ satisfying the relations

$$\bar{\alpha}x = y, \quad \bar{\alpha}y = x\bar{\alpha} = y\bar{\alpha} = x^2 = y^2 = xy = yx = 0.$$

Hence $V(R, L)$ has a K-linear basis given by $\{\bar{\alpha}^n, x, y\}_{n \in \mathbb{N}}$. The source and target maps are both the inclusion of R into $V(R, L)$. Hence one can also see directly that $V(R, L)$ admits no antipode: S would satisfy $S(x) = x$, $S(y) = y$ and one would have $y = S(y) = S(\bar{\alpha}x) = S(x)S(\bar{\alpha}) = xS(\bar{\alpha})$, but there is no element $z \in V(R, L)$ such that $y = xz$.

ACKNOWLEDGMENT

A.R. thanks José Figueroa O'Farrill for his encouragement, and Gwyn Bellamy for remarks about differential operators.

FUNDING

U.K. acknowledges support by the EPSRC grant (No. EP/J012718/1) "Hopf algebroids and Operads" and the Polish Government Grants 2012/06/M/ST1/00169; A.R. is funded by an EPSRC DTA grant.

REFERENCES

[1] Böhm, G. (2009). *Hopf Algebroids*. Handbook of Algebra, Vol. 6. Amsterdam: Elsevier/North-Holland, pp. 173–235.

[2] Böhm, G., Szlachányi, K. (2004). Hopf algebroids with bijective antipodes: Axioms, integrals, and duals. *J. Algebra* 274(2):708–750.

[3] Evens, S., Lu, J.-H., Weinstein, A. (1999). Transverse measures, the modular class and a cohomology pairing for Lie algebroids. *Quart. J. Math. Oxford Ser. (2)* 50(200): 417–436.

[4] Huebschmann, J. (1998). Lie–Rinehart algebras, Gerstenhaber algebras and Batalin-Vilkovisky algebras. *Ann. Inst. Fourier (Grenoble)* 48(2):425–440.

[5] Huebschmann, J. (2008). *The Universal Hopf Algebra Associated with a Hopf-Lie-Rinehart Algebra*. Preprint arXiv:0802.3836.

[6] Kowalzig, N., Krähmer, U. (2010). Duality and products in algebraic (co)homology theories. *J. Algebra* 323(7):2063–2081.

[7] Kowalzig, N. (2009). Hopf algebroids and their cyclic theory. Ph.D. thesis, Universiteit Utrecht.

[8] Kowalzig, N., Posthuma, H. (2011). The cyclic theory of Hopf algebroids. *J. Noncommut. Geom.* 5(3):423–476.

[9] Lu, J.-H. (1996). Hopf algebroids and quantum groupoids. *Internat. J. Math.* 7(1): 47–70.

[10] Mackenzie, K. (1987). Lie groupoids and Lie algebroids in differential geometry. *London Mathematical Society Lecture Note Series*, Vol. 124. Cambridge: Cambridge University Press.

[11] Moerdijk, I., Mrčun, J. (2010). On the universal enveloping algebra of a Lie algebroid. *Proc. Amer. Math. Soc.* 138(9):3135–3145.

[12] Pradines, J. (1967). Théorie de Lie pour les groupoïdes différentiables. Calcul différentiel dans la catégorie des groupoïdes infinitésimaux. *C. R. Acad. Sci. Paris Sér. A-B*, 264:A245–A248.

[13] Rinehart, G. S. (1963). Differential forms on general commutative algebras. *Trans. Amer. Math. Soc.* 108:195–222.

[14] Schauenburg, P. (2000). Duals and doubles of quantum groupoids ($\times_R$-Hopf algebras). *New trends in Hopf algebra theory (La Falda)*, *Contemp. Math.* 267:273–299. Providence, RI: Amer. Math. Soc., 2000.

[15] Takeuchi, M. (1977). Groups of algebras over $A \otimes \overline{A}$. *J. Math. Soc. Japan* 29(3): 459–492.

[16] Xu, P. (2001). Quantum groupoids. *Comm. Math. Phys.* 216(3):539–581.