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## PAPER

## Divergence of an orbital-angular-momentum-carrying beam upon propagation

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 Miles J Padgett<sup>1</sup>, Filippo M Miatto<sup>2</sup>, Martin P J Lavery<sup>1</sup>, Anton Zeilinger<sup>3,4</sup> and Robert W Boyd<sup>1,2,5</sup>
<sup>1</sup> School of Physics and Astronomy, University of Glasgow, Glasgow G12 8QQ, UK

<sup>2</sup> Department of Physics, University of Ottawa, 150 Louis Pasteur, Ottawa, Ontario, K1N 6N5, Canada

<sup>3</sup> Institute for Quantum Optics and Quantum Information (IQOQI), Austrian Academy of Sciences, Boltzmanngasse 3, A-1090 Vienna, Austria

<sup>4</sup> Vienna Center for Quantum Science and Technology, Faculty of Physics, University of Vienna, Boltzmanngasse 5, A-1090 Vienna, Austria

<sup>5</sup> The Institute of Optics, University of Rochester, Rochester, NY 14627, USA

 E-mail: [miles.padgett@glasgow.ac.uk](mailto:miles.padgett@glasgow.ac.uk)

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### Abstract

There is recent interest in the use of light beams carrying orbital angular momentum (OAM) for creating multiple channels within free-space optical communication systems. One crucial issue is that, for a given beam size at the transmitter, the beam divergence angle increases with increasing OAM. Therefore the larger the value of OAM, the larger the aperture required at the receiving optical system if the efficiency of detection is to be maintained. Confusion exists as to whether this divergence scales linearly with, or with the square root of, the beam's OAM. We clarify how both these scaling laws are valid, depending upon whether it is the radius of the waist of the beam's Gaussian term or the radius of rms intensity of the beam that is kept constant while varying the OAM.

### 1. Introduction

Over the past 20 years there has been a growing interest in the orbital angular momentum (OAM) of light, which is carried by any optical beam possessing helical phase fronts [1]. For phase fronts described by an azimuthally dependent phase factor  $\exp(i\ell\phi)$ , the OAM is equivalent to  $\ell\hbar$  per photon [2]. This OAM of a light beam can be derived in a number of ways, but for our purposes it can be seen to arise from the local direction of the Poynting vector. At all points within the beam, the Poynting vector is perpendicular to the phase front and hence possesses an azimuthal momentum component [3]. Throughout this paper we consider only the case of free-space propagation of light beam. As confirmed by measurement [4], the helical phase fronts of an OAM-carrying beam lead to the local Poynting vector being skewed with respect to the beam axis by an angle  $\beta$ , which is a function of  $r$ , the radial distance from the beam axis:

$$\beta = \frac{|\ell|}{k_0 r}, \quad (1)$$

where  $k_0 = 2\pi/\lambda$ . One analytically simple and widely used form of helically phased beams is the Laguerre–Gaussian (LG) modes which have a field amplitude given as the product of a Gaussian term with beam waist  $w_0$  (defined as the radius at which the electric field of the Gaussian term is lower by a factor  $1/e$ ), with a generalized Laguerre polynomial with indices  $|\ell|$  and  $p$ . The radial index  $p$  gives the number of off-axis nodes in the radial direction.

In this work we restrict ourselves to the small numerical aperture case where the size of any spatial features in the modes or their superpositions is significantly larger than the optical wavelength. For  $p = 0$  and  $\ell = 0$  one has the fundamental Gaussian mode. For  $p = 0$  and  $\ell > 0$  the LG modes are single-ringed annular modes with an azimuthal phase term  $\exp(i\ell\phi)$  and a radius of maximum optical intensity,  $r(I_{\max})$ , given as [3]

$$r(I_{\max}) = \sqrt{\frac{|\ell|}{2}} w(z), \quad (2)$$

where  $w(z) = w_0 \sqrt{1 + z^2/z_R^2}$  and  $z_R = \frac{1}{2}kw_0^2$  is the Rayleigh range (here  $k = 2\pi/\lambda$ ). One notes that for high OAM values the size of the LG mode is much larger than the beam waist. The restriction of our study to the paraxial regime also implies that  $w(z) \gg \lambda$  and hence that  $\beta$  (see equation (1)) remains small for all radii at which the beam has a non-negligible intensity.

In terms of their complex amplitudes, LG beams with different mode indices are orthogonal to each other and this leads to their potential use in communication applications where the orthogonality of the modes allows for multiple states within a channel or independent channels. A simple optical communication system based on OAM was demonstrated over 10 years ago [5], but recently the idea has been revisited with reports of both high data rates [6] and outdoor demonstrations [7]. One issue of critical importance is the degree to which the use of OAM-carrying beams require the use of larger aperture optics. Confusion exists as to whether the divergence and hence the far-field size of a beam carrying OAM scales linearly or with the square root of its angular momentum.

## 2. Divergence of an OAM carrying beam

The the geometrically calculated skew angle of the Pointing vector calculated with respect to the beam axis given by equation (1) might be thought to approximate the angular divergence of the beam. However, an additional contribution to the divergence that has to be considered is the normal diffractive spreading arising from the finite beam diameter. Rather than the beam waist  $w_0$ , or the radius of maximum intensity  $r(I_{\max})$ , the divergence of a light beam is governed by the standard deviation of its spatial distribution,  $r_{\text{rms}}$  [8, 9]. The calculation of these quantities is straightforward. We start from the definition of a LG mode which gives the intensity distribution for the lowest-order radial mode ( $p=0$ ) as

$$I_\ell(r, \phi, z) = \frac{2}{w(z)^2 \pi |\ell|!} \left( \frac{\sqrt{2} r}{w(z)} \right)^{2|\ell|} \exp\left(-\frac{2r^2}{w(z)^2}\right), \quad (3)$$

which is normalized such that  $\int I_\ell(r, \phi, z) r dr d\phi = 1$ . From this distribution, we can calculate the radial position of the maximum of intensity, by solving the equation  $\partial I_\ell / \partial r = 0$ , obtaining (as anticipated in equation (2) [3])

$$r(I_{\max}) = \sqrt{\frac{|\ell|}{2}} w(z). \quad (4)$$

However, as  $r(I_{\max})$  is zero for  $\ell = 0$ , clearly one cannot generally use this quantity to derive the angular spread of the beam as it propagates. As has been reported previously, a better quantity on which to base a calculation of the divergence of a light beam is the standard deviation of the spatial distribution of the beam,  $r_{\text{rms}}$ . This radius is a function of  $z$  and it is given by the square root of the radial variance of the intensity distribution [10]:

$$r_{\text{rms}}(z) = \sqrt{2\pi \int_0^\infty r^2 I_\ell(r, \phi, z) r dr} \quad (5)$$

$$= \sqrt{\frac{|\ell| + 1}{2}} w(z). \quad (6)$$

In the plane of the beam waist,  $z=0$ , this radius is

$$r_{\text{rms}}(0) = \sqrt{\frac{|\ell| + 1}{2}} w_0. \quad (7)$$

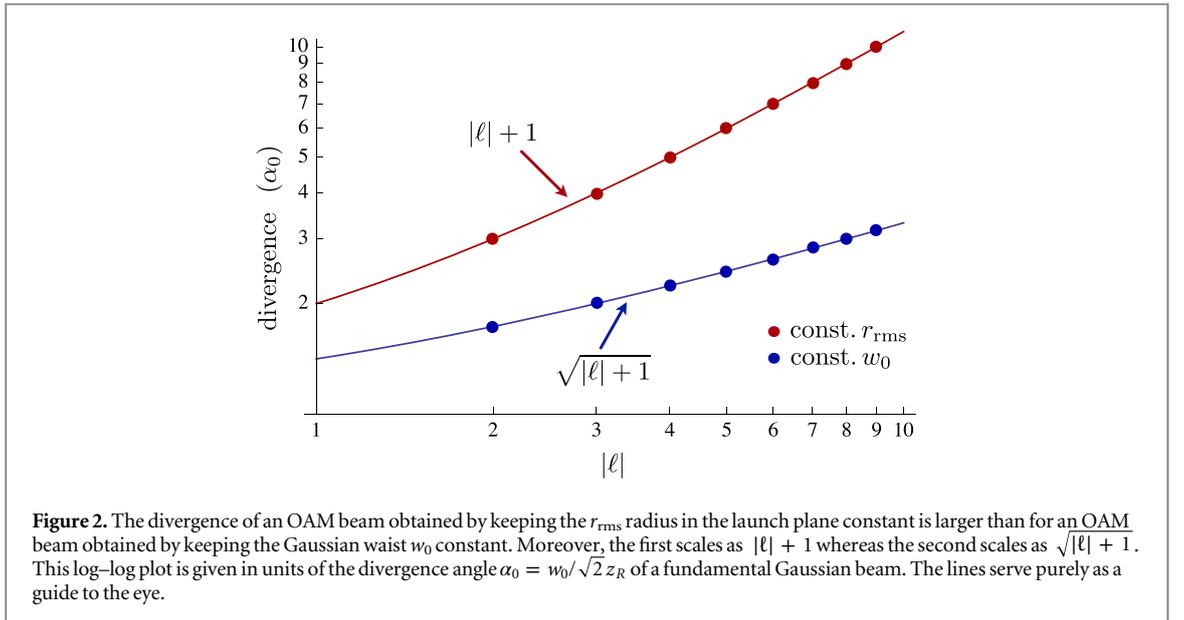
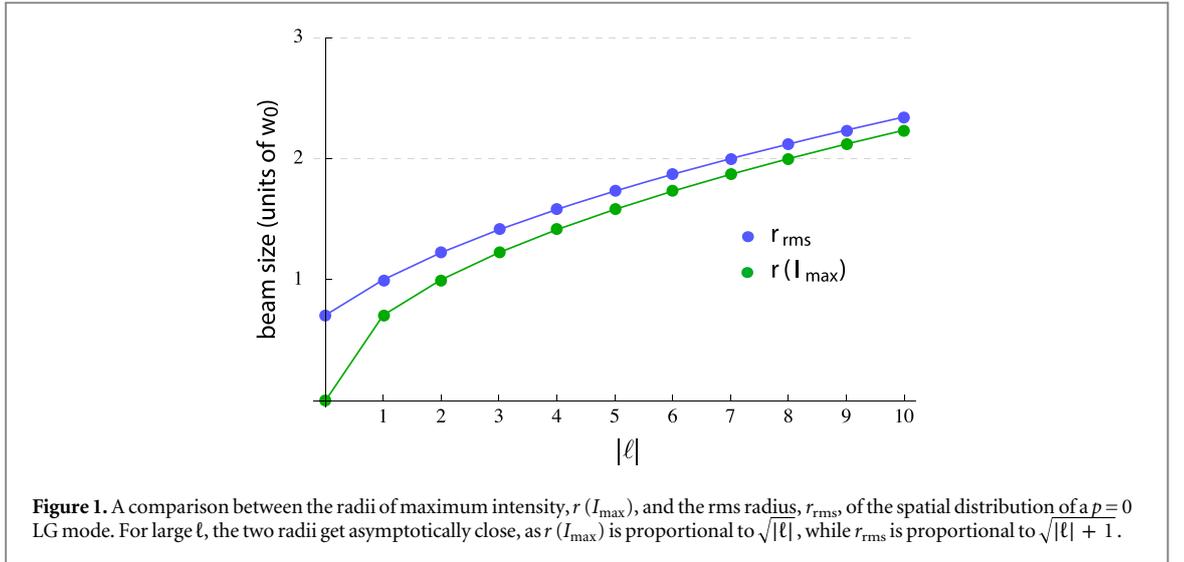
In any given plane, figure 1 shows the  $\ell$  dependence of  $r(I_{\max})$  and  $r_{\text{rms}}$ . One notes that

$$\frac{r_{\text{rms}}}{r(I_{\max})} = \sqrt{\frac{|\ell| + 1}{|\ell|}}, \quad (8)$$

which tends to 1 as  $|\ell|$  increases, so these two radii become equal for large  $\ell$ , see figure 1.

Given the radius  $r_{\text{rms}}(z)$  as a function of propagation distance,  $z$ , we can calculate the corresponding divergence angle of a  $p=0$  LG mode,  $\alpha_\ell$ , to be

$$\alpha_\ell(z) = \arctan\left(\frac{\partial r_{\text{rms}}(z)}{\partial z}\right) \quad (9)$$



$$= \arctan\left(\sqrt{\frac{|\ell| + 1}{2}} \frac{w_0^2}{z_R^2} \frac{z}{w(z)}\right). \quad (10)$$

This divergence angle is a function of  $z$ , but it asymptotically reaches a limiting value (when the beam is sufficiently far from the waist). Within the paraxial regime, for small  $\alpha_\ell$ , we can equate the tangent of the angle with the angle and for  $\lim_{z \rightarrow \infty} z/w(z) = z_R/w_0$ . Hence in the far field, the divergence angle of a  $p=0$  LG mode is given by

$$\alpha_\ell = \sqrt{\frac{|\ell| + 1}{2}} \frac{2}{k_0 w_0}. \quad (11)$$

Equation (11) is a convenient form for expressing the  $\ell$  dependence of the beam divergence while holding  $w_0$  of the LG mode constant, revealing the approximate square root scaling of the far-field beam size; see figure 2.

Alternatively the divergence angle of the beam can be written in terms of other quantities; for instance, using equation (7), one can instead express the  $\ell$  dependence of the beam divergence for a constant  $r_{\text{rms}}(0)$  as

$$\alpha_\ell = \frac{|\ell| + 1}{k_0 r_{\text{rms}}(0)}. \quad (12)$$

Under this condition, of a fixed  $r_{\text{rms}}(0)$ , we see that the beam divergence scales linearly with  $\ell$ , see figure 2. Note also for a fixed  $r_{\text{rms}}(0)$  that  $w_0$  itself becomes a function of  $\ell$  and scales as  $(|\ell| + 1)^{-1/2}$ .

### 3. Discussion

Typical of a practical optical system is that the aperture of the optics is fixed by design, with a limiting radius  $R$ . Since the radius of maximum intensity of an LG beam can be much larger than  $w_0$ , in order to design a system capable of using a wide range of different OAM modes one must choose a waist  $w_0 \ll R$ .

One example of an approach that produces modes of a fixed beam waist is the cylindrical-lens mode converter that converts any incoming Hermite Gaussian mode into the corresponding LG mode with the same beam waist [11]. When using a mode converter of this type to produce the LG modes one should observe a square root scaling of the far-field beam divergence, equation (11), i.e. a scaling approximately proportional to  $\sqrt{|\ell|}$ .

A more frequently used and flexible method of producing beams carrying OAM is the illumination of a diffractive optical element (computer generated hologram) with an expanded Gaussian mode from a conventional laser. If the diffractive element is a spatial light modulator (or similar) then the displayed kinoform can be updated to change the beam type allowing a switching of the system between different OAM states. Most common of such kinoforms is a forked diffraction grating that produces a helically phased beam in the first diffraction order [12]. In the plane of the diffractive element, the beam has the required helical phase structure, but possesses the Gaussian intensity distribution of the illuminating beam. This beam corresponds to a superposition of different LG modes, all of the same OAM  $\ell$ , but with various values of  $p$ . The precise  $p$ -weighting of this superposition is a complicated function of both  $|\ell|$  and the value of beam waist  $w_0$  chosen to perform the decomposition [13]. An obvious choice of beam waist for the decomposition is the one which maximizes the amount of power in the  $p = 0$  mode, as referred to above this optimum beam waist scales  $(|\ell| + 1)^{-1/2}$ . Therefore rather than the resulting beam having a particular  $|\ell|$ -dependent beam waist, the  $|\ell|$ -invariant quantity is simply the size of the beam incident on the diffraction grating and hence the system has a fixed  $r_{\text{rms}}(0)$ . Therefore when using a forked diffraction grating as the beam generation component within an optical system to produce LG-like modes one should observe the approximately linear  $|\ell|$ -scaling of the far-field beam divergence, equation (12). This linear scaling of the beam size as a function of  $|\ell|$  has previously been noted within optical tweezers and microscope systems [14].

We note that the reasoning we have followed above is consistent with an understanding of the scaling in terms of the spatial resolution of an optical system. For example, if one considers a transmitted beam comprising a superposition of  $\pm\ell$  modes, then the resulting beam profile is an annular ring of  $2|\ell|$  maxima, or ‘petals’. In the far field the spacing between these petals cannot be smaller than the resolving power of the optical system. Since the number of petals scales with  $|\ell|$ , this implies that the beam size and hence the beam divergence must also scale with  $|\ell|$ . A square root scaling of the beam divergence  $\alpha_\ell$  and corresponding decrease in petal separation would seemingly be in conflict with this simple resolution argument. However, a square root scaling requires a constant beam waist  $w_0$ , which implies a beam size  $r_{\text{rms}}(0)$  and hence a minimum transmission aperture that must increase with the square root of  $|\ell|$ . This increasing aperture itself supports a higher spatial resolution, meaning that the reduced spacing of the resulting petals remains compatible with the resolution limit.

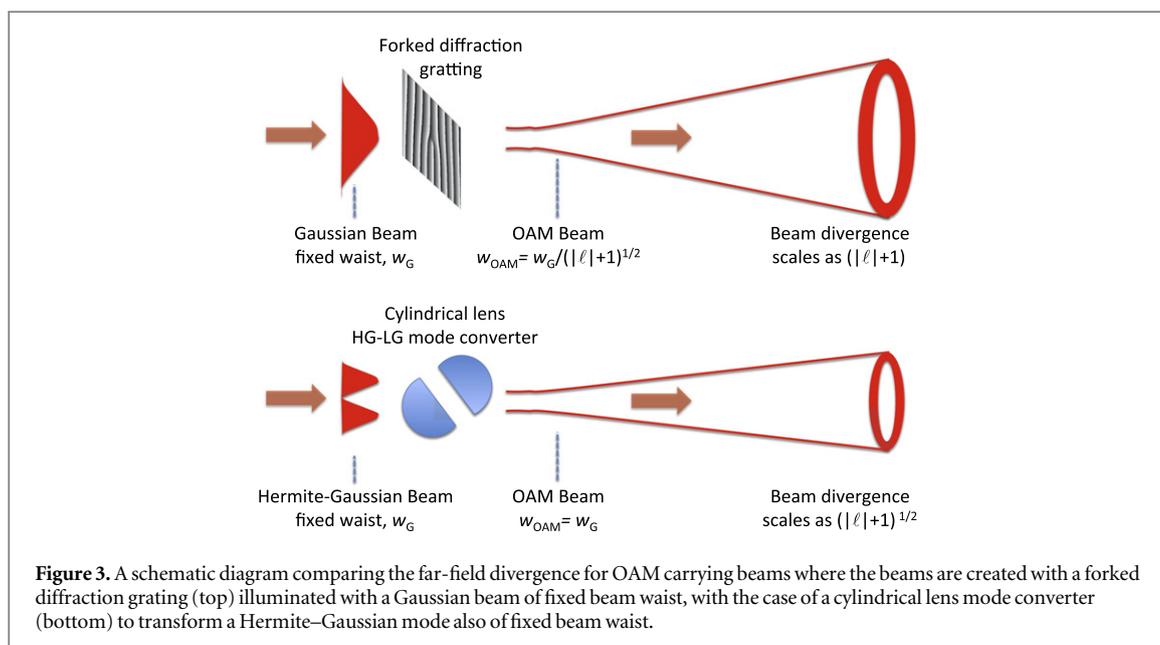
### 4. Conclusions

In conclusion, we have explained how optical systems exploiting the propagation of light beams carrying OAM can produce a far-field beam divergence that scales either with the square root of the OAM or linearly with the OAM, see figure 3. The square root scaling applies to systems where the beam waist of the Gaussian term,  $w_0$ , is held constant for all values of  $\ell$ , such as the case where the beams are produced using a cylindrical lens mode converter [11]. The linear scaling of the beam divergence applies to systems where the rms radius of the launch beam is held constant, such as the case where the beam is produced by a simple forked diffraction grating implemented on a spatial light modulator illuminated with a Gaussian beam of fixed beam<sup>6</sup>.

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<sup>6</sup> One notes that more sophisticated hologram designs can produce arbitrary combinations of LG modes with any values of  $\ell$ ,  $p$  and  $w_0$  (see for instance M Dennis *et al Nat. Phys.* 6, 118 (2010)), albeit at greatly reduced efficiency compared to the simple fork design that is widely used.



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