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Vibration of beams with piezoelectric inclusions

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Abstract

A mathematical model for the vibration of beams with piezoelectric inclusions is presented. The piezoelectric inclusion in a non-piezoelectric matrix (host beam) is analyzed as two inhomogeneous inclusion problems, elastic and dielectric, by using Eshelby's equivalent inclusion method. The natural frequency of the beam is determined from the variational principle in Rayleigh quotient form, which is expressed as functions of the elastic strain energy and dielectric energy of the piezoelectric inclusion. The Euler–Bernoulli beam theory and Rayleigh–Ritz approximation technique are used in the present analysis. In addition, a parametric study is conducted to investigate the influence of the energies due to piezoelectric coupling on the natural frequency of the beam.

Keywords: Vibration; Beam; Piezoelectric inclusions; Natural frequency; Eshelby's equivalent inclusion method

1. Introduction

Smart structures are systems that incorporate particular functions such as sensing, processing and actuation. They have the ability to sense certain stimuli and respond in a controlled manner (Chung, 2002). Smart structures are important because of their relevance to structural health monitoring, structural vibration control and transportation engineering. A primary focus in the research of smart structures is the use of piezoelectric materials, since these materials can function both as sensors and actuators (Tani et al., 1998).

A major problem in the dynamic operation of structures is undesirable vibrations (Mackerle, 2003). This is why vibration control and active damping are among the most studied areas using smart materials and structures (Cao et al., 1999; Chee et al., 1998). Extensive research has been done on the vibration control and suppression of structures using piezoelectric materials, as evident from numerous review articles (see for example Ahmadian and DeGuilio, 2001; Irschik, 2002; Rao and Sunar, 1994; Sunar and Rao, 1999; Wetherhold and Aldraihem, 2001). Many mathematical models for laminates and structures with piezoelectric sensors and/or

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actuators have been presented in the literature, and reviews of these models have been presented by Alzahrani and Alghamdi (2003), Chee et al. (1998), Gopinathan et al. (2000), Chopra (2002) and Saravanos and Heyliger (1999). Brief summaries on the computational models for composite laminates with piezoelectric sensors and actuators can also be found in Mota Soares et al. (2000) and Reddy (1999). A bibliographical review on the finite element models for the analysis and simulation of smart materials and structures have been presented by Mackerle (2003).

An analysis of these reviews indicates that piezoceramic materials are widely used as sensors and/or actuators. They are either in the form of patches or lamina. The piezoelectric patches are either bonded to or embedded within the structures, whereas the piezoelectric lamina are stacked together with a substrate laminae to form a piezoelectric composite laminate (Chee et al., 1998). However, there are several factors that limit the use of piezoceramic materials, such as their brittle nature and low tensile strength, therefore limiting their ability to conform to curved shapes, and the large add-on mass associated with using typical lead-based piezoceramic (Williams et al., 2002). The use of arrays of piezoelectric sensors and actuators embedded within the structure would remedy the above mentioned restrictions. Due to their small size, these sensors/actuators have the flexibility to conform to curved shapes, and they add little weight to the structure (Badcock and Birt, 2000). In addition, these piezoelectric sensors and actuators can be tailored to achieve a particular smart structure design.

Owing to the small size of the piezoelectric sensors and actuators relative to the size of the host structure, these sensors/actuators can be analyzed as inclusions in a non-piezoelectric matrix (host structure) by using a micromechanics approach. Fan and Qin (1995) analyzed a piezoelectric sensor embedded in a non-piezoelectric elastic matrix by using Eshelby's equivalent inclusion method (Eshelby, 1957; Mura, 1987). The piezoelectric problem was decoupled into an elastic inclusion problem and a dielectric inclusion problem connected by some eigenstrain and eigenelectric field. Jiang et al. (1997, 1999) analyzed the piezoelectric inclusion in a non-piezoelectric matrix by using the Green's function technique.

Krommer and Irschik (1999), Irschik et al. (1998) and Irschik and Ziegler (2001) analyzed the piezoelectric actuation for vibration and shape control of structures as an eigenstrain actuation. An eigenstrain technique was presented by Alghamdi and Dasgupta (1993a,b, 2000, 2001) for the vibration of beams with embedded arrays of piezoelectric sensors and actuators. The embedded sensors and actuators were analyzed as piezoelectric ellipsoidal inclusion in an infinite matrix (host beam) by using Eshelby's equivalent inclusion method. Using the variational principle in Rayleigh quotient form, they formulated an equation for the natural frequency of the beam, which was expressed as functions of the elastic strain energy and dielectric energy of the beam. However, the piezoelectric inclusions were analyzed as elastic inclusions only, thereby neglecting the dielectric effects of the piezoelectric inclusions. The influence of the mechanical–electrical coupling of the piezoelectric sensors on the natural frequency was also neglected in their analyses.

In this research, a mathematical model for the vibration of beams with embedded arrays of piezoelectric sensors and actuators is presented. The piezoelectric sensors and actuators are analyzed as inhomogeneous ellipsoidal inclusions in a non-piezoelectric matrix (host beam) by using Eshelby's equivalent inclusion method (Eshelby, 1957; Mura, 1987). The formulation for the piezoelectric inclusion problem is decoupled into two equivalent inclusion problems, an elastic problem and a dielectric problem. An equation for the natural frequency of the beam is determined using the variational principle in Rayleigh quotient form, which is expressed as functions of the elastic strain energy and dielectric energy of the piezoelectric inclusions. These energies are derived using Mura's formulation for inhomogeneous inclusions. The Euler–Bernoulli beam theory and Rayleigh–Ritz approximation technique are used in the present analysis. In addition, the influences of the energies due to the electromechanical coupling of the actuators and the mechanical–electrical coupling of the sensors on the natural frequency of the beam are studied.

The research is presented as follows. First, the mathematical modeling is presented, which begins with the formulation of the equation for the natural frequency of a piezoelectric body. This is followed the analysis of a non-piezoelectric matrix with piezoelectric ellipsoidal inclusions using Eshelby's equivalent inclusion method. The energies of the piezoelectric inclusions are then formulated, and an explicit solution for the natural frequency of a beam with piezoelectric inclusions is obtained. Next, using the mathematical model presented, the influence of the energies due to piezoelectric coupling on the natural frequency of the beam is studied.

2. Mathematical modeling

2.1. Variational principle in Rayleigh quotient form

Let Ω be a region occupied by a piezoelectric body and S be boundary surface of Ω. The constitutive relations for a linear piezoelectric material are

$$\sigma_{ij} = C_{ijmn}\varepsilon_{mn} - e_{nij}E_n \quad \text{in } \Omega \tag{1}$$

$$D_i = e_{imn}\varepsilon_{mn} + \kappa_{in}E_n \quad \text{in } \Omega \tag{2}$$

where σ_{ij}, ε_{ij}, E_i and D_i are the stress tensor, strain tensor, electric field vector and the electric displacement vector, respectively. C_{ijmn}, e_{nij} and κ_{in} are the elastic stiffness tensor, the piezoelectric tensor and permittivity tensor, respectively. The strain and electric field are derivable from the mechanical and electric potential as

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{3}$$

$$E_i = -\phi_{,i} \tag{4}$$

where u_i is the displacement and ϕ is the electric potential. For the time-harmonic free vibration of a piezoelectric body with circular frequency ω, the governing equations and boundary conditions in rectangular Cartesian coordinates are (Tiersten, 1969)

$$\sigma_{ij,j} = (C_{ijmn}u_{m,n} + e_{nij}\phi_{,n})_{,j} = \rho\omega^2 u_i \quad \text{in } \Omega \tag{5}$$

$$D_{i,i} = (e_{imn}u_{m,n} - \kappa_{in}\phi_{,n})_{,i} = 0 \quad \text{in } \Omega \tag{6}$$

$$u_i = 0 \quad \text{on } S \tag{7}$$

$$\sigma_{ij}n_j = (C_{ijmn}u_{m,n} + e_{nij}\phi_{,n})n_j = 0 \quad \text{on } S \tag{8}$$

$$\phi = 0 \quad \text{on } S \tag{9}$$

$$D_i n_i = (e_{imn}u_{m,n} - \kappa_{in}\phi_{,n})n_i = 0 \quad \text{on } S \tag{10}$$

where n_j is the outward pointing unit normal vector. Conventional indicial notation is utilized where repeated subscripts are summed over the range 1–3 and the comma denotes partial differentiation. In the MKS system, the variables of piezoelectricity have the following units (Dunn and Taya, 1993):

$$[\sigma_{ij}] = \mathrm{N\,m^{-2}},\quad [D_i] = \mathrm{N\,V^{-1}\,m^{-1}},\quad [\varepsilon_{ij}] = \mathrm{mm^{-1}},\quad [E_i] = \mathrm{V\,m^{-1}},\quad [u_i] = \mathrm{m}$$

$$[C_{ijmn}] = \mathrm{N\,m^{-2}},\quad [e_{imn}] = \mathrm{N\,V^{-1}\,m^{-1}} = \mathrm{C\,m^{-2}},\quad [\kappa_{in}] = \mathrm{N\,V^{-2}} = \mathrm{C^2\,N^{-1}\,m^{-2}},\quad [\phi] = \mathrm{V}$$

Eqs. (5) and (6) are the Euler equations and Eqs. (7)–(10) are the boundary conditions of the following stationary expression G (Eernisse, 1967)

$$\begin{aligned} G &= \frac{1}{2}\int_V (\sigma_{ij}\varepsilon_{ij} + E_iD_i - \rho\omega^2 u_iu_i)\,\mathrm{d}V - \int_S (\sigma_{ij}n_ju_i - \phi D_in_i)\,\mathrm{d}S \\ &= \int_V \left(\frac{1}{2}C_{ijmn}u_{i,j}u_{m,n} - \frac{1}{2}\kappa_{in}\phi_{,i}\phi_{,n} + e_{imn}u_{m,n}\phi_{,i} - \frac{1}{2}\rho\omega^2 u_iu_i\right)\mathrm{d}V - \int_S u_i(C_{ijmn}u_{m,n} + e_{nij}\phi_{,n})n_j\,\mathrm{d}S \\ &\quad + \int_S \phi(e_{imn}u_{m,n} - \kappa^*_{in}\phi_{,n})n_i\,\mathrm{d}S \end{aligned} \tag{11}$$

For the stationary expression G, the variation is zero ($\delta G = 0$), then $G = 0$ (Eernisse, 1967). The Rayleigh quotient for ω^2 can be written as

$$\begin{aligned} \omega^2 &= \frac{\int_V \left(\frac{1}{2}C_{ijmn}u_{i,j}u_{m,n} - \frac{1}{2}\kappa_{in}\phi_{,i}\phi_{,n} + e_{imn}u_{m,n}\phi_{,i}\right)\mathrm{d}V}{\int_V \frac{1}{2}\rho u_iu_i\,\mathrm{d}V} \\ &\quad + \frac{\int_S u_i(C_{ijmn}u_{m,n} + e_{nij}\phi_{,n})n_j\,\mathrm{d}S - \int_S \phi(e_{imn}u_{m,n} - \kappa_{in}\phi_{,n})n_i\,\mathrm{d}S}{\int_V \frac{1}{2}\rho u_iu_i\,\mathrm{d}V} \end{aligned} \tag{12}$$

By adding and subtracting $\frac{1}{2}\kappa_{in}\phi_{,i}\phi_{,n}$ to the integrand of the numerator in the first term on the RHS of Eq. (12) and applying integration by parts and the divergence theorem, the Rayleigh quotient for stationary solutions can be expressed as

$$\omega^2 = \frac{\int_V \left(\frac{1}{2}C_{ijmn}u_{i,j}u_{m,n} + \frac{1}{2}\kappa_{in}\phi_{,i}\phi_{,n}\right)\mathrm{d}V}{\int_V \frac{1}{2}\rho u_i u_i \,\mathrm{d}V} = \frac{\frac{1}{2}\int_V C_{ijmn}\varepsilon_{ij}\varepsilon_{mn}\,\mathrm{d}V + \frac{1}{2}\int_V \kappa_{in}E_iE_n\,\mathrm{d}V}{\frac{1}{2}\int_V \rho u_i u_i \,\mathrm{d}V} \tag{13}$$

which was obtained by Eernisse (1967) and Yang and Batra (1994).

The numerator in Eq. (13) is the internal energy of the system, which is the sum of the elastic strain energy and the dielectric energy. These energies will be analyzed using a micromechanics approach.

2.2. Non-piezoelectric matrix with piezoelectric inclusions

According to Mura (1987), eigenstrain is a generic name for non-elastic strains resulting from thermal expansion, phase transformation, initial strains, plastic strains and misfit strains. An inclusion is a sub-domain Ω in a domain D, where eigenstrain is prescribed in Ω and is zero in the matrix $D-\Omega$. The elastic moduli of the inclusion are assumed to be the same as the matrix. When the sub-domain Ω with prescribed eigenstrain has different elastic moduli with the matrix, Ω is called an inhomogeneous inclusion.

With the above definitions, let us consider an infinite non-piezoelectric matrix $D-\Omega$ with elastic moduli C_{ijmn} containing a piezoelectric ellipsoidal inclusion, perfectly bonded to the matrix, with domain Ω and elastic moduli C^*_{ijmn} (Fig. 1). Since the elastic moduli of the piezoelectrics are different to that of the host beam, they are analyzed as inhomogeneous inclusions by using Eshelby equivalent inclusion method (Eshelby, 1957; Mura, 1987).

The constitutive equation for the piezoelectric inclusion is given in Eqs. (1) and (2) and is rewritten as follows:

$$\sigma_{ij} = C^*_{ijmn}\varepsilon_{mn} - e_{nij}E_n \quad \text{in } \Omega \tag{14}$$

$$D_i = e_{imn}\varepsilon_{mn} + \kappa^*_{in}E_n \quad \text{in } \Omega \tag{15}$$

Since there is no electromechanical coupling in the matrix, the constitutive equation for the matrix is

$$\sigma_{ij} = C_{ijmn}\varepsilon_{mn} \quad \text{in } D-\Omega \tag{16}$$

$$D_i = \kappa_{in}E_n \text{ in } \quad D-\Omega \tag{17}$$

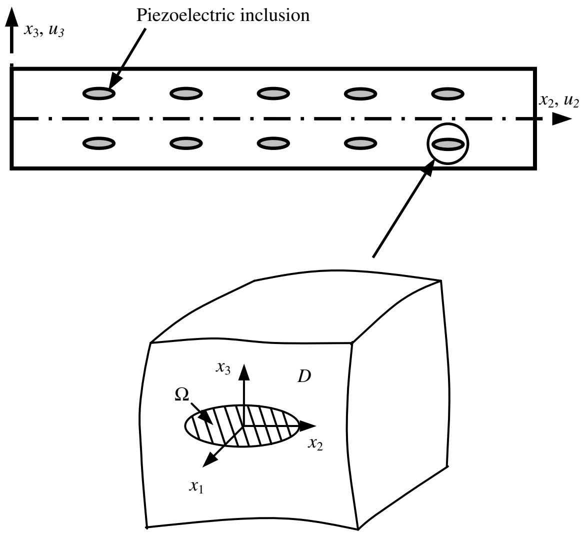

Fig. 1. A beam with piezoelectric inclusions.

The superscript '*' refers to the material property of the piezoelectric inclusions. The assumption of no electro-mechanical interaction in the matrix material partially decouples the original piezoelectric inclusion problem into two equivalent inclusion problems, namely, an elastic inclusion problem and dielectric inclusion problem (Fan and Qin, 1995).

For the elastic inclusion problem, the Hooke's law is expressed as (Eshelby, 1957; Mura, 1987)

$$\sigma_{ij}^0 + \sigma_{ij} = C_{ijmn}^*(\varepsilon_{mn}^0 + \varepsilon_{mn}) - e_{nij}(E_n^0 + E_n) \tag{18}$$

or

$$\sigma_{ij}^0 + \sigma_{ij} = C_{ijmn}^*(\varepsilon_{mn}^0 + \varepsilon_{mn} - \varepsilon_{mn}^p) \tag{19}$$

where

$$C_{ijmn}^*\varepsilon_{mn}^p = e_{nij}(E_n^0 + E_n) \tag{20}$$

and where σ_{ij}^0 is the uniform far field loading, ε_{mn}^0 is the strain corresponding to the far field loading, σ_{ij} and ε_{mn} are the disturbance stress and strains due to the presence of inhomogeneity, respectively, and ε_{mn}^p is the eigenstrain due to the electromechanical coupling of the piezoelectric actuator or the eigenstrain actuation (Irschik et al., 1998; Irschik and Ziegler, 2001; Krommer and Irschik, 1999).

According to the equivalent inclusion method, one can convert the inhomogeneous inclusion to an equivalent inclusion with the elastic constant of the matrix and a uniform fictitious eigenstrain ε_{mn}^*. With this concept, Eq. (19) can be written as

$$\sigma_{ij}^0 + \sigma_{ij} = C_{ijmn}(\varepsilon_{mn}^0 + \varepsilon_{mn} - \varepsilon_{mn}^p - \varepsilon_{mn}^*) \tag{21}$$

Equating Eqs. (19) and (21), we have

$$C_{ijmn}^*(\varepsilon_{mn}^0 + \varepsilon_{mn} - \varepsilon_{mn}^p) = C_{ijmn}(\varepsilon_{mn}^0 + \varepsilon_{mn} - \varepsilon_{mn}^{**}) \tag{22}$$

where the total eigenstrain ε_{mn}^{**} is defined as

$$\varepsilon_{mn}^{**} = \varepsilon_{mn}^p + \varepsilon_{mn}^* \tag{23}$$

The strain disturbance ε_{mn} can be related to the total eigenstrain by the elastic Eshelby tensor S_{mnab} as

$$\varepsilon_{mn} = S_{mnab}\varepsilon_{ab}^{**} \tag{24}$$

The elastic Eshelby tensor S_{mnab} is only a function of the matrix Poisson's ratio and the ellipsoidal aspect ratios. The components of the Eshelby tensor are well documented in Mura (1987). Substituting Eq. (24) in Eq. (22) and rearranging, ε_{mn}^{**} can be expressed as

$$\varepsilon_{mn}^{**} = -[(C_{ijkl}^* - C_{ijkl})S_{klmn} + C_{ijmn}]^{-1}[(C_{ijrs}^* - C_{ijrs})\varepsilon_{rs}^0 - C_{ijrs}^*\varepsilon_{rs}^p] \tag{25}$$

The above equivalent inclusion method is repeated here to for the dielectric inclusion problem. Eq. (15) is rewritten as

$$D_i^0 + D_i = \kappa_{in}^*(E_n^0 + E_n - E_n^p) \tag{26}$$

where

$$-\kappa_{in}^*E_n^p = e_{imn}(\varepsilon_{mn} + \varepsilon_{mn}^0) \tag{27}$$

where D_i^0 and E_n^0 are the known far fields, and E_n^p is caused by the mechanical–electrical coupling. Similarly with the elastic inclusion problem, we obtain the relationship

$$\kappa_{in}^*(E_n^0 + E_n - E_n^p) = \kappa_{in}(E_n^0 + E_n - E_n^p - E_n^*) \tag{28}$$

where E_n^* is the eigenelectric field. By defining a total eigenelectric field E_n^{**} as

$$E_n^{**} = E_n^p + E_n^* \tag{29}$$

and an electric Eshelby tensor, s_{nb}, the electric field disturbance E_n can be written as

$$E_n = s_{nb}(E_b^p + E_b^*) = s_{nb}E_b^{**} \tag{30}$$

The electric Eshelby tensor, s_{nb}, is only a function of the ellipsoidal aspect ratio and the components are presented in Hatta and Taya (1986) and Fan and Qin (1995). Substituting Eq. (30) in Eq. (28) and rearranging, E_n^{**} can be expressed as

$$E_n^{**} = [-(\kappa_{im}^* - \kappa_{im})s_{mn} - \kappa_{in}]^{-1}[(\kappa_{ik}^* - \kappa_{ik})E_k^0 - \kappa_{ik}^* E_k^p] \tag{31}$$

2.3. Energies of piezoelectric inclusion

2.3.1. Elastic strain energy

The elastic strain energy of the piezoelectric inclusion is obtained using the method presented in Mura (1987) for inhomogeneous inclusions. When a body D, containing an inhomogeneous inclusion Ω with eigenstrain ε_{ij}^p, is subjected to an external force F_i, the elastic strain energy is

$$W_e = \frac{1}{2}\int_D (\sigma_{ij}^0 + \sigma_{ij})(u_{i,j}^0 + u_{i,j} - \varepsilon_{ij}^p)\,\mathrm{d}D \tag{32}$$

which can be expressed as

$$W_e = \frac{1}{2}\int_D \sigma_{ij}^0 \varepsilon_{ij}^0\,\mathrm{d}D + \frac{1}{2}\int_\Omega \sigma_{ij}^0 \varepsilon_{ij}^*\,\mathrm{d}D - \frac{1}{2}\int_\Omega \sigma_{ij}\varepsilon_{ij}^p\,\mathrm{d}D \tag{33}$$

where ε_{ij}^* can be determined from

$$C_{ijmn}^*(\varepsilon_{mn}^0 + S_{mnab}\varepsilon_{ab}^*) = C_{ijmn}(\varepsilon_{mn}^0 + S_{mnab}\varepsilon_{ab}^* - \varepsilon_{mn}^*) \tag{34}$$

and σ_{ij} can be determined from

$$\sigma_{ij} = C_{ijmn}^*(S_{mnab}\varepsilon_{ab}^{**} - \varepsilon_{ab}^p) = C_{ijmn}(S_{mnab}\varepsilon_{ab}^{**} - \varepsilon_{ab}^{**}) \tag{35}$$

The details of the formulation can be found in Mura (1987).

2.3.2. Dielectric energy

Similarly, the dielectric energy can be determined using Mura's method for elastic inhomogeneous inclusions. The dielectric energy is given by

$$W_d = \frac{1}{2}\int_\Omega (D_i^0 + D_i)(E_i^0 + E_i - E_i^p)\,\mathrm{d}D \tag{36}$$

where

$$D_i^0 = -\kappa_{in}^*\phi_{,n}^0 = \kappa_{in}^* E_n^0 \tag{37}$$

and

$$D_i^0 + D_i = \kappa_{in}^*(E_n^0 + E_n - E_n^p) \tag{38}$$

By integration by parts and applying the divergence theorem, Gauss' law of electrostatic ($D_{i,i} = 0$ in D) and electrical boundary conditions ($D_i n_i = 0$ on S), we obtain

$$\int_\Omega D_i(-\phi_{,i}^0 - \phi_{,i})\,\mathrm{d}D = \int_\Omega D_i(E_i^0 + E_i)\,\mathrm{d}D = 0 \tag{39}$$

Furthermore, we have

$$D_i^0(E_i - E_i^p) = \kappa_{in}^* E_n^0(E_i - E_i^p) = E_n^0\kappa_{in}^*(E_i - E_i^p) = D_i E_i^0 \tag{40}$$

By integration by parts and applying the divergence theorem

$$\int_{\Omega} D_i E_i^0 \, \mathrm{d}D = \int_{\Omega} D_i(-\phi_{,i}^0) \, \mathrm{d}D = \int_S D_i(-\phi^0) n_i \, \mathrm{d}S - \int_{\Omega} D_{i,i}(-\phi^0) \, \mathrm{d}D = 0 \tag{41}$$

From the above Eqs. (36)–(41), the dielectric energy of the piezoelectric inclusion is determined by

$$W_d = \frac{1}{2} \int_{\Omega} D_i^0 E_i^0 \, \mathrm{d}D - \frac{1}{2} \int_{\Omega} D_i E_i^p \, \mathrm{d}D \tag{42}$$

where D_i can be determined from

$$D_i = \kappa_{in}^* (s_{na} E_a^{**} - E_i^p) \tag{43}$$

2.4. Natural frequency of a beam with piezoelectric inclusions

Fig. 1 shows a beam with piezoelectric inclusions. The piezoelectric inclusions in the one of the rows are considered to act as sensors and the other row, as actuators. Bending the beam induces strain in the sensors and it produces an output, which is used in a closed-loop constant-feedback-gain control circuit to activate the corresponding actuators. The actuation induces strain along the length of the beam causing an effect opposite to that caused by the initial bending. This results in a stiffening of the beam and an accompanying increase in the natural frequency (Alghamdi and Dasgupta, 1993a,b).

In order for the analyses presented in Sections 2.2 and 2.3 to be applicable for the present problem, the following conditions must be satisfied: the piezoelectric sensors/actuators are far enough below the free surface of the beam and the distance between neighboring sensors/actuators is large enough so that the interactions among the sensors/actuators can be neglected. To be able to satisfy the above conditions, the volume of the piezoelectric materials must be small relative to the volume of the host beam, which can be about less than five percent of the volume of the host (Alghamdi and Dasgupta, 1993a,b).

The Rayleigh quotient in Eq. (13) presents an explicit solution to estimate the natural frequency of a piezoelectric body. We extend the use Eq. (13) for a non-piezoelectric beam with piezoelectric inclusions, which can be written as

$$\omega^2 = \frac{\frac{1}{2}\int_D C_{ijmn}\varepsilon_{ij}^0\varepsilon_{mn}^0 \, \mathrm{d}D + \frac{1}{2}\int_{\Omega} C_{ijmn}\varepsilon_{ij}^*\varepsilon_{mn}^0 \, \mathrm{d}D - \frac{1}{2}\int_{\Omega} C_{ijmn}^*\varepsilon_{ij}^p(S_{mnab}\varepsilon_{ab}^{**} - \varepsilon_{mn}^p) \, \mathrm{d}D}{\frac{1}{2}\int_V \rho u_i u_i \, \mathrm{d}V} + \frac{\frac{1}{2}\int_{\Omega} \kappa_{in}^* E_i^0 E_n^0 \, \mathrm{d}D - \frac{1}{2}\int_{\Omega} \kappa_{in}^* E_i^p(s_{na}E_a^{**} - E_n^p) \, \mathrm{d}D}{\frac{1}{2}\int_V \rho u_i u_i \, \mathrm{d}V} \tag{44}$$

It should be noted that the elastic strain energy, which consists of first three terms in the numerator on the right-hand side in Eq. (44), was based Mura's method for inhomogeneous inclusion and is different to that presented by Alghamdi and Dasgupta (1993a,b, 2000, 2001). The third term in the numerator represents the elastic strain energy due to the electromechanical coupling of the piezoelectric actuators, which will be referred to as *eigenstrain actuation energy*. The fifth term represents the dielectric energy due to the mechanical–electrical coupling of the piezoelectric sensors, which was not included in the analyses of Alghamdi and Dasgupta (1993a,b, 2000, 2001). This energy will be referred to as *eigenelectric energy*. For an unactuated beam, the energy consist of the first two terms in the numerator.

In order to perform the integration in Eq. (44), all that remains now is to assume explicit representations for the applied flexural strain field and the eigenstrain actuation. We make further assumptions:

a. The Euler–Bernoulli beam theory and plane stress assumption are used.
b. Each of the embedded piezoelectric sensors/actuators is assumed to be a piezoelectric cylinder with elliptical cross-section, whose polarization is oriented along the thickness of the beam (x_3-axis).
c. The actuator will have both eigenstrain actuation and fictitious eigenstrain, whereas the sensor will only have fictitious eigenstrain.

d. Both sensors and actuators will have real eigenelectric field due to the direct piezoelectric effect (applied stress on piezoelectric materials induces electric polarization). However, for the sensors, the fictitious eigenelectric field is zero, since the electric field is applied to the piezoelectric actuator (see Eq. (31)).

Using the Rayleigh–Ritz technique, for a simply supported beam, the transverse displacement function is given by

$$u_3 = \sum_{n=1}^{\infty} a_n \sin(\omega_n t) \sin\left(\frac{n\pi x_2}{L}\right) \tag{45}$$

where the x_2-axis is oriented along the length of the beam, u_3 is the transverse deflection in the x_3 direction, ω_n and a_n are the natural frequency and amplitude of the nth mode. Only the fundamental frequency ($n = 1$) is of interest in this study. In view of the Euler–Bernoulli beam theory and plane stress assumptions, the only non-zero term in the bending strain field ε^0_{ij} is ε^0_2 and is determined by

$$\varepsilon^0_2 = -z\frac{\partial^2 u_3}{\partial x_2^2} = z\frac{\pi^2}{L^2} a_1 \sin(\omega_1 t) \sin\left(\frac{\pi x_2}{L}\right) \tag{46}$$

where z is the distance of the piezoelectric device from the neutral axis of the beam.

The only non-zero component of the actuation voltage vector is E^0_3 and is proportional to the bending strain ε^0_2. The eigenstrain actuation ε^p_{ij} is determined from Eq. (20)

$$\varepsilon^p_{mn} = d_{kmn} E^0_k \tag{47}$$

where

$$d_{kmn} = [C^*_{ijmn}]^{-1} e_{kij} \tag{48}$$

and where d_{kmn} is the free-expansion of the piezoelectric actuator for a unit applied electric field. The non-zero terms of the eigenstrain actuation ε^p_{mn} expressed in contracted Voight notation is

$$\varepsilon^p_q = d_{3q} E^0_3 \quad (q = 1\text{–}6) \tag{49}$$

The eigenelectric field due to the mechanical–electrical coupling E^p_k is determined from Eq. (27)

$$E^p_k = -[\kappa^*_{ik}]^{-1} e_{imn}(\varepsilon_{mn} + \varepsilon^0_{mn}) \tag{50}$$

and the non-zero terms are

$$E^p_3 = -\frac{e_{3q}}{\kappa^*_{33}}(\varepsilon_q + \varepsilon^0_q) \quad (q = 1\text{–}6) \tag{51}$$

3. Results and discussion

This section presents the results obtained using the mathematical model described in Section 2 to study the influence of the energies of the piezoelectric inclusion on the natural frequencies of a simply supported beam. For the purpose of verification, the present results are compared with the results of Alghamdi and Dasgupta (1993b). The host beam is made of Alplex material and piezoelectric sensors/actuators are made of PZT-5H. Table 1 presents the properties of Alplex and PZT-5H, which is poled along the length of the beam (x_2-axis). The dimensions of the beam and the piezoelectric inclusions are shown in Fig. 2, where the inclusions are located between the surface and neutral axis of the beam ($z = e/4$). The normalized fundamental frequency

Table 1
Properties of the host beam and the piezoelectric material poled along x_2-axis

	E (GPa)	υ	d_{21} (10^{-12} m/V)	d_{22} (10^{-12} m/V)	d_{16} (10^{-12} m/V)	κ_{11} (10^{-10} C/V m)	κ_{22} (10^{-10} C/V m)
PZT-5H	64	0.39	−274	593	741	150	66
Alplex	3.5	0.35	–	–	–	–	–

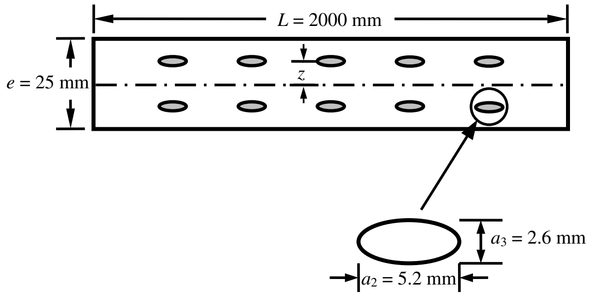

Fig. 2. Dimensions of the beam and the piezoelectric inclusions.

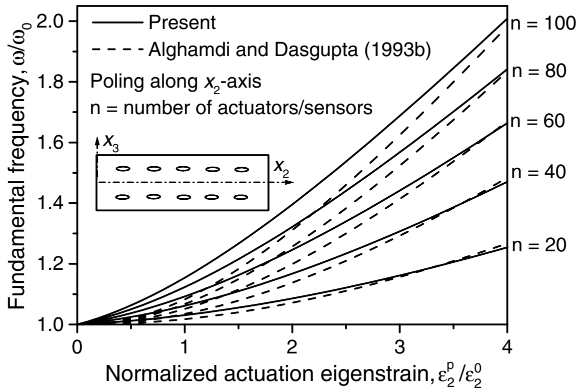

Fig. 3. Comparison of predicted fundamental frequencies, ω/ω_0, with the theoretical results of Alghamdi and Dasgupta (1993b).

ω/ω_0 as a function of the normalized eigenstrain actuation $\varepsilon_2^p/\varepsilon_2^0$ is shown in Fig. 3, where ω_0 is the fundamental frequency of the beam without the actuation effect. The number of actuators, n, is identical to the number of sensors. It can be seen that the present results compared well with the results of Alghamdi and Dasgupta (1993b).

Figs. 4–8 show the influence of the eigenstrain actuation energy on the natural frequency of the beam. The influence of this energy is investigated by considering the difference between the frequencies when all the

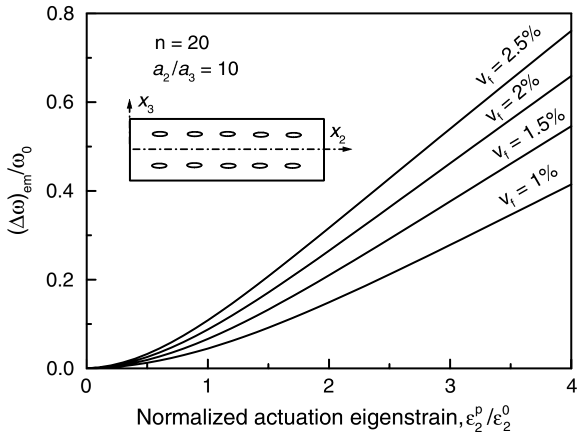

Fig. 4. Changes in the natural frequency due to energy of the electromechanical coupling, $(\Delta\omega)_{em}/\omega_0$, with eigenstrain actuation, $\varepsilon_2^p/\varepsilon_2^0$, for various volume fraction, v_f.

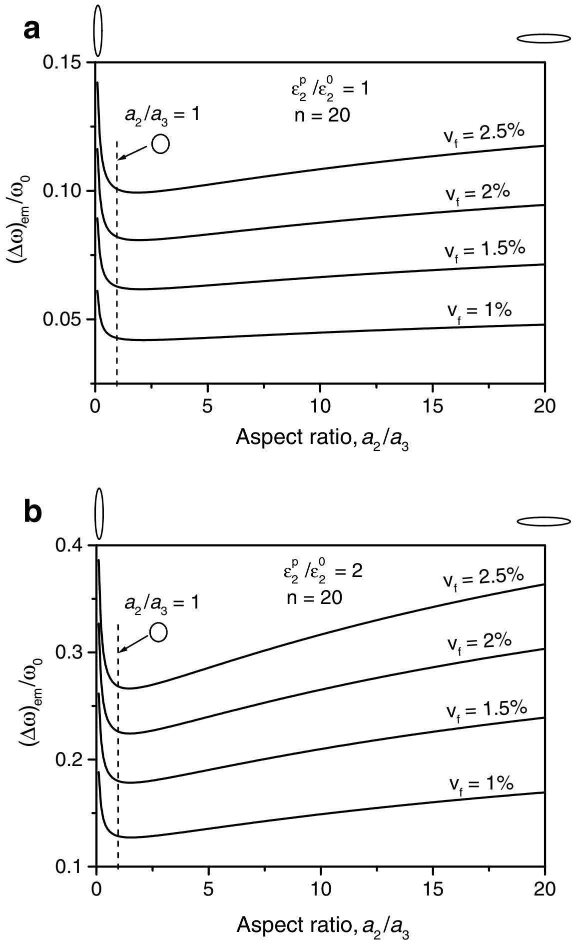

Fig. 5. Changes in the natural frequency due to energy of the electromechanical coupling, $(\Delta\omega)_{em}/\omega_0$, with aspect ratio, a_2/a_3, for various volume fraction, v_f: (a) $\varepsilon_2^p/\varepsilon_2^0 = 1$; (b) $\varepsilon_2^p/\varepsilon_2^0 = 2$.

energy terms are included in Eq. (44) and when the eigenstrain actuation energy (third term in the numerator) is neglected, where $(\Delta\omega)_{em}$ is the difference between the two frequencies. In these results the piezoelectric material is poled along the thickness of the beam (x_3-axis), where the piezoelectric properties are given in Table 2.

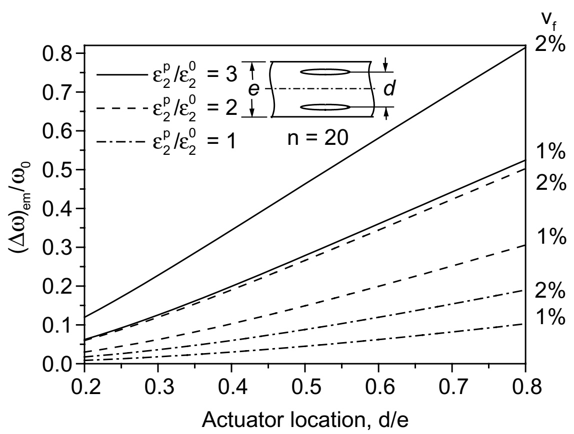

Fig. 6. Changes in the natural frequency due to energy of the electromechanical coupling, $(\Delta\omega)_{em}/\omega_0$, with actuator location, d/e.

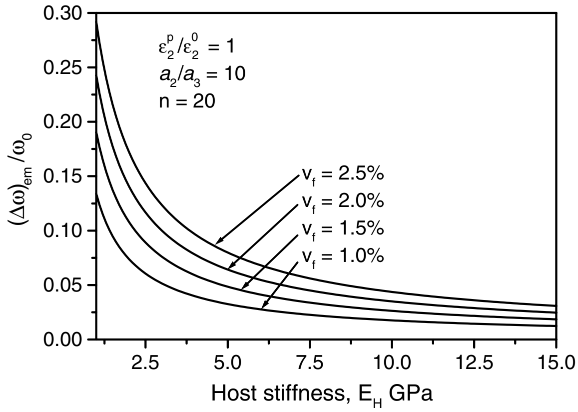

Fig. 7. Changes in the natural frequency due to energy of the electromechanical coupling, $(\Delta\omega)_{em}/\omega_0$, with the host stiffness for various volume fraction, v_f.

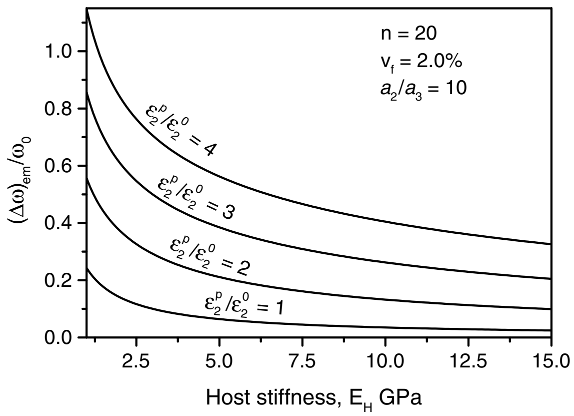

Fig. 8. Changes in the natural frequency due to energy of the electromechanical coupling, $(\Delta\omega)_{em}/\omega_0$, with the host stiffness for various eigenstrain actuation, $\varepsilon_2^p/\varepsilon_2^0$.

Table 2
Properties of the the piezoelectric material poled along x_3-axis

	E (GPa)	υ	d_{31} (10^{-12} m/V)	d_{33} (10^{-12} m/V)	d_{15} (10^{-12} m/V)	κ_{11} (10^{-10} C/V m)	κ_{33} (10^{-10} C/V m)
PZT-5H	64	0.39	−274	593	741	150	66

The influence of eigenstrain actuation energy increases with increasing eigenstrain actuation $\varepsilon_2^p/\varepsilon_2^0$ and increasing piezoelectric volume fraction, v_f (Fig. 4). Flat piezoelectric sensors/actuators further increase the influence of the actuation energy (Fig. 5), which becomes more obvious for higher actuation $\varepsilon_2^p/\varepsilon_2^0$.

In the results presented in Figs. 4 and 5, the piezoelectric sensors and actuators are located between the surface and neutral axis of the beam ($z = e/4$). Fig. 6 shows the influence of the eigenstrain actuation energy with the actuator location d/e. The influence increases as the actuator is located nearer to the beam surface and increases further with increasing v_f and $\varepsilon_2^p/\varepsilon_2^0$.

The variation of $(\Delta\omega)_{em}/\omega_0$ with the stiffness of the host beam E_H is shown in Figs. 7 and 8, where the sensors and actuators are located between the neutral axis and the beam surface. It is observed that $(\Delta\omega)_{em}/\omega_0$ decreases drastically up to about $E_H = 2.5$ GPa, it then decreases slowly up to about $E_H = 7$ GPa, after which it displays an almost constant frequency. This can be explained by the decreasing influence of the eigenstrain

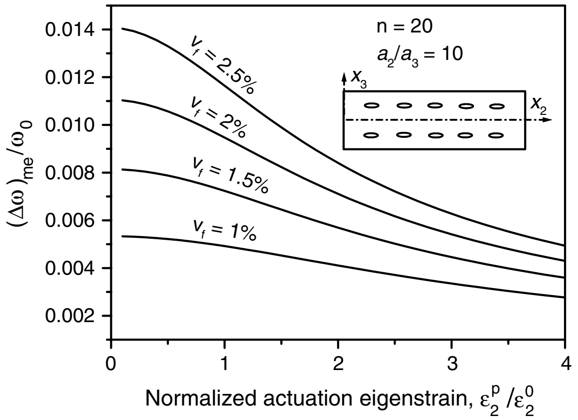

Fig. 9. Changes in the natural frequency due to energy of the mechanical–electrical coupling, $(\Delta\omega)_{me}/\omega_0$, with eigenstrain actuation, $\varepsilon_2^p/\varepsilon_2^0$, for various volume fraction, v_f.

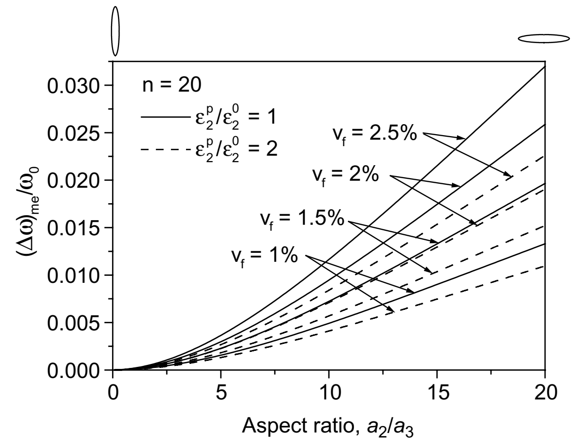

Fig. 10. Changes in the natural frequency due to energy of the mechanical–electrical coupling, $(\Delta\omega)_{me}/\omega_0$, with the aspect ratio, a_2/a_3.

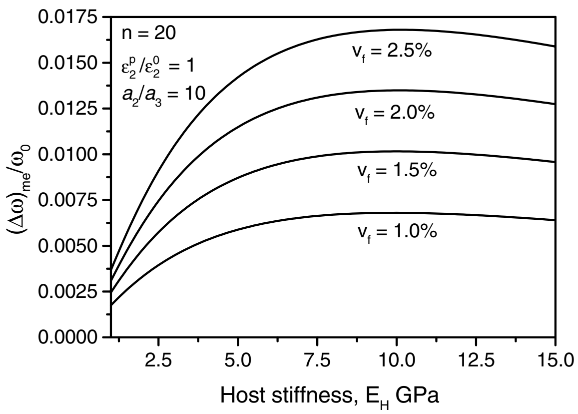

Fig. 11. Changes in the natural frequency due to energy of the mechanical–electrical coupling, $(\Delta\omega)_{me}/\omega_0$, with the host stiffness for various volume fraction, v_f.

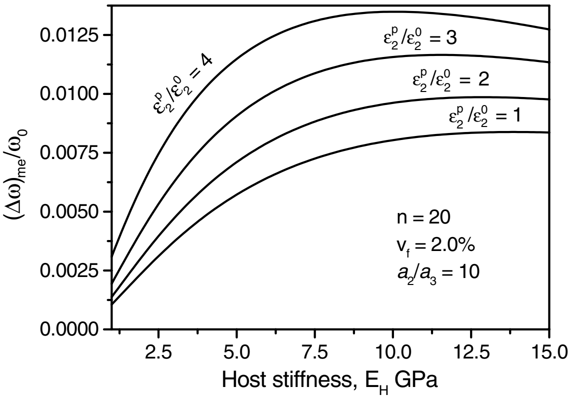

Fig. 12. Changes in the natural frequency due to energy of the mechanical–electrical coupling, $(\Delta\omega)_{me}/\omega_0$, with the host stiffness for various eigenstrain actuation, $\varepsilon_2^p/\varepsilon_2^0$.

actuation energy as the host stiffness increases. Furthermore, the $(\Delta\omega)_{em}/\omega_0$ decreases more for higher volume fraction v_f (Fig. 7) and higher $\varepsilon_2^p/\varepsilon_2^0$ (Fig. 8).

Figs. 9–12 show the influence of the eigenelectric energy on the natural frequency of the beam, where $(\Delta\omega)_{me}$ is the difference between the frequencies when all the energy terms are included in Eq. (44) and when the eigenelectric energy (fifth term in the numerator) is neglected. Fig. 9 shows that the influence of the eigenelectric energy on the natural frequency of the beam decreases with increasing eigenstrain actuation $\varepsilon_2^p/\varepsilon_2^0$. This influence increases as aspect ratio a_2/a_3 increases (Fig. 10), however, it decreases with increase eigenstrain actuation, $\varepsilon_2^p/\varepsilon_2^0$.

In Figs. 11 and 12, influence of the eigenelectric energy on the natural frequency increases with increasing host stiffness and increases with increasing volume fraction v_f (Fig. 11) and increasing actuation eigenstrain $\varepsilon_2^p/\varepsilon_2^0$. However, the maximum influence is less than 2% of ω_0.

4. Conclusions

Through the use of the variational principle in Rayleigh quotient form, Eshelby's equivalent inclusion method and Mura's formulation for the energies of inhomogeneous inclusions, an explicit solution for the natural frequency of a beam with piezoelectric inclusions was obtained. A parametric study was conducted to investigate the influence of the energies due to the electromechanical coupling of the actuators and the mechanical–electrical coupling of the sensors on the natural frequency. Based on this study the following conclusions are made:

1. The influence of the eigenstrain actuation energy on the natural frequency of the beam is significant, while the influence of the eigenelectric energy is less significant.
2. The influence of the eigenstrain actuation energy on the natural frequency of the beam increases with increasing eigenstrain actuation and decreases with increasing host stiffness, in contrast, the influence of the eigenelectric energy decreases with increasing eigenstrain actuation and increasing host stiffness.
3. Flat piezoelectric materials increase the influence of the eigenstrain actuation energy, whereas, piezoelectric materials with high aspect ratio increase the influence of the eigenelectric energy.

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